MINIMAX ESTIMATION OF LOCATION PARAMETERS FOR SPHERICALLY SYMMETRIC UNIMODAL DISTRIBUTIONS

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ABSTRACT

When sampling from a p-dimensional spherically symmetric unimodal (s.s.u.) distribution about an unknown parameter θ, with invariant loss $L(\delta,\theta)$, the usual estimator of θ is the best invariant procedure which is inadmissible for $p \geq 3$. With respect to quadratic loss and general quadratic loss, we find explicit minimax estimators which are better than the best invariant procedure. Specifically, when the loss is general quadratic loss given by $L(\delta,\theta) = (\delta-\theta)'D(\delta-\theta)$ where D is a $p \times p$ positive definite matrix, one main result, for one observation, X, on a multivariate s.s.u. distribution about θ, presents a class of minimax estimators whose risks dominate the risk of X, provided $p \geq 3$ and $\text{trace}D \geq 2d_L$ where d_L is the maximum eigenvalue of D. This class is given by $\delta_{a,r}(X) = (1-a(r(\|X\|^2)/\|X\|^2))X$ where $0 \leq r(\cdot) \leq 1$, $r(\|X\|^2)$ is non-decreasing, $r(\|X\|^2)/\|X\|^2$ is non-increasing, and $0 \leq a \leq (c_0/E_0(\|X\|^{-2}))((\text{trace}D/d_L)-2)$, where $c_0 = 2p/((p+2)(p-2))$ when $p \geq 4$ and $c_0 = .96$ when $p = 3$.

1. Introduction. When sampling from a p-dimensional spherically symmetric unimodal (s.s.u.) distribution about an unknown parameter θ, with invariant loss $L(\delta,\theta)$, the usual estimator of θ is the best invariant procedure. For $p \geq 3$ it has long been known that the best invariant procedure is inadmissible with respect to a large class of loss functions. However, except in certain cases explicit estimators which are better have not been found. In this paper we will present explicit minimax estimators which are better than the best invariant procedure when the loss is one of the following:

Quadratic (Sum of Squared Errors) Loss

$$L(\delta,\theta) = ||\delta - \theta||^2 = \sum_{i=1}^{p} (\delta_i - \theta_i)^2 \tag{1.1}$$

where $\delta = [\delta_1, \delta_2, \ldots, \delta_p]'$ and $\theta = [\theta_1, \theta_2, \ldots, \theta_p]'$

or General Quadratic Loss

$$L(\delta,\theta) = (\delta-\theta)' D(\delta-\theta) \tag{1.2}$$

where D is a p x p positive definite symmetric matrix.

The best known spherically symmetric unimodal distribution about θ is the p-variate normal distribution with mean vector θ and covariance matrix the identity $\left(MVN(\theta, Ip)\right)$. Stein [16] in 1955 investigated the question of the admissibility of the best invariant procedure, $\bar{X}$, when sampling from a $MVN(\theta, Ip)$ distribution with respect to quadratic loss (1.1). The admissibility of the best invariant procedure for one dimension was well known due to some of the works that preceded Stein's [4, 10, 11]. Although Stein's original objective was to prove the admissibility of $\bar{X}$ for $p \geq 2$, his final results stated that $\bar{X}$ is admissible for $p = 2$ and inadmissible for $p \geq 3$. In addition, in 1961 James and Stein [12] once again considered this problem and proved that the inadmissibility of $\bar{X}$ was not exclusive to the case of quadratic loss but was true for any loss function $L(\delta,\theta) = F(||\delta-\theta||)$ where F has a bounded derivative, is continuously differentiable and concave. It was also stated that these results not only apply to the normal distribution but are true for other location parameter family distributions when certain fourth moment conditions are satisfied.

This work seemed to indicate that the inadmissibility of the best invariant procedure when $p \geq 3$ could be generalized to a larger class of distributions.

Brown [6, 7], in 1965 and 1966 respectively, considered this general location parameter estimation problem and proved that under mild assumptions on the loss function, the best invariant procedure is admissible in one and two dimensions and inadmissible in three or more dimensions.

The work of Stein and Brown motivates a new problem, namely that of finding explicit estimators which are better than the best invariant procedure when sampling from a location parameter family. Heretofore, such estimators have only been found for the normal and "mixtures" of normals distributions.

In their 1961 paper, James and Stein, in addition to the admissibility results stated, also presented explicit estimators for the mean vector of a $MVN(\theta, Ip)$ distribution which are better than X, where X is one observation on the distribution. The estimators considered were of the form

(1.3) $\delta_a(X) = (1 - a/||X||^2)X$

It was proven that for $p \geq 3$ and $0 \leq a \leq 2(p-2) = 2/E_0(\|X\|^{-2})$ estimators of the form (1.3) are minimax with respect to the loss given by (1.1). These estimators are at least as good as the best invariant procedure X which is known to be minimax. In addition, when $0 < a < 2/E_0(\|X\|^{-2})$, δ_a is in fact better than X.

In 1964, Baranchik [2] found a class of minimax estimators for θ which includes the James Stein class, when X is one observation on a $MVN(\theta, Ip)$ distribution and the loss is given by (1.1). It was proven that $\delta_r(X) = \left(1 - (p-2)\left(r(\|X\|^2/2)/\|X\|^2\right)\right)X$ is minimax provided $p \geq 3$, $r(\|X\|^2/2)$ is a non-decreasing function of $\|X\|$ and $0 \leq r(\cdot) \leq 2$.

For a long time after this, explicit estimators which were better than the best invariant procedure were only available for the mean vector of a multivariate normal distribution. In 1974, Strawderman [17] found explicit estimators which are better than the best invariant procedure when sampling from certain s.s.u. distributions about θ. He considered the problem of estimating the location parameter θ when X is a p x 1 random vector with a distribution which has a density (with respect to Lebesgue measure) of the form

$$\int (2\pi\sigma^2)^{-p/2} \exp\left(-(\|X-\theta\|^2/2\sigma^2)\right) \, dG(\sigma)$$

where $G(\cdot)$ is a known cdf on $(0,\infty)$, i.e. variance "mixtures" of normals. Although this class is not the whole class of s.s.u. location parameter families it does contain "thick" tailed as well as "thin" tailed distributions by a suitable choice of $G(\cdot)$. He proved that when $p \geq 3$, $\delta_{a,r}(X) = \left(1 - a(r(\|X\|^2)/\|X\|^2)\right)$, is minimax provided $r(\|X\|^2)$ is non-decreasing, $r(\|X\|^2)/\|X\|^2$ is non-increasing, $0 \leq a \leq 2/E_0(\|X\|^{-2})$ and the loss is sum of squared errors (1.1). As in previous cases, the minimaxity of these estimators was proven by showing they are at least as good or better than X.

P. K. Bhattechaya [3] and later Bock [5] considered estimation problems for the MVN(θ,Ip) distribution when the loss is general quadratic loss (1.2). In particular, Bock in 1975 extended the results of James and Stein and Baranchik to this case. Provided trace $D > 2d_L$, where d_L = maximum eigenvalue of D, she was able to find explicit minimax estimators of the mean vector θ.

When X is one observation on a p-dimensional spherically symmetric unimodal distribution about θ we produce analogous results to those of James and Stein, Baranchik, Strawderman and Bock.

We begin with one observation X on a p-dimensional uniform distribution over a sphere, $(||X-\theta||^2 \leq R^2)$, with known radius R. Hence, the density of X is given by

$$f_\theta(x) = c(R)I_S(x,R)$$

where $S = \left((x,R): \; ||X-\theta||^2 \leq R^2\right)$, R is known,

$$I_S(x,R) = \begin{cases} 1 \text{ if } (x,R) \in S \\ 0 \text{ if } (x,R) \notin S \end{cases} \tag{1.4}$$

and

$$c(R) = 1/\int I_S(x,R)dx.$$

By first considering the estimators of James and Stein given by (1.3) and later those of Baranchik and Bock, we explicitly find classes of minimax estimators with respect to losses (1.1) and (1.2).

As in previous works, we show that the risk of the best invariant procedure X (which is minimax) is dominated by the risks of these new procedures. In addition, all estimators given in our classes of minimax estimators which are not identically X are better than X.

As in the work of Strawderman we broaden this problem by extending our work for the uniform distribution to the problem of estimating the location parameter when the distribution under consideration has a density which is a "mixture" of uniforms, i.e. X is one observation on a distribution which has a density (with respect to Lebesgue measure) of the form

$$g(\|x-\theta\|) = \int c(R)I_S(x,R)dF(R)$$

where $c(R)$, S and $I_S(x,R)$ are defined by (1.4) and $F(\cdot)$ is a known cdf on $(0,\infty)$. We find explicit minimax estimators which are better than X when $p \geq 3$.

Specifically, for $p \geq 4$ we prove that with respect to loss (1.1) $\delta_{a,r}(X) = \left(1 - a(r(\|X\|^2)/\|X\|^2)\right)X$ is better than X for $0 < a \leq (2p/(p+2))/E_0(\|X\|^{-2})$ when the distribution is a "mixture" of spherical uniforms, provided $0 \leq r(\cdot) \leq 1$, $r(\|X\|^2)$ is non-decreasing and $r(\|X\|^2)/\|X\|^2$ is non-increasing. When $p = 3$, the results are slightly different from those we obtain for $p \geq 4$. However, minimax estimators of the form given for $p \geq 4$ can be gotten when $p = 3$ by adjusting the constant a (the upper limit on a).

A p x 1 random vector X is said to have a s.s.u. distribution about θ if the density g of X with respect to Lebesgue measure is a non-increasing function of $\|X-\theta\|$. It is known (although proof is given in this paper) that such a density can be written as a "mixture" of uniform distributions. Hence, by mixing normals Strawderman obtains explicit minimax estimators for some s.s.u. distributions about θ and we, by mixing uniforms, obtain explicit minimax estimators for all s.s.u. distributions about θ.

We close this introduction by presenting an ordered outline of this paper. Section 2 contains analagous results to those of James and Stein and Baranchik for the spherical uniform distribution when the loss is (1.1). Section 3 is an extension of these results to one observation on a s.s.u. distribution about θ. An extension of the results given in Sections 2 and 3 for general quadratic loss (1.2) is given in Section 4. In Section 5, we make some statements about the multiple observation case as well as the usefulness and benefits of using the improved estimators. Lastly, Section 6 is an Appendix containing many useful integral expressions as well as other facts used throughout this paper.

2. Minimax estimators of the location parameter of a p-dimensional spherical uniform distribution with respect to quadratic loss. We consider the problem of estimating the location parameter θ of a p-dimensional ($p \geq 3$) spherical uniform distribution.

Definition 2.1. A p x 1 random vector X is said to have a p-dimensional spherical uniform distribution with location parameter θ ($X \sim U\{||X-\theta||^2 \leq R^2\}$) if the density of X with respect to Lebesgue measure is

$$f_\theta(x) = c(R) I_S(x,R) \text{ where}$$
$$S = \left((x,R): \ ||x-\theta||^2 \leq R^2\right), \ R \text{ is known,}$$
$$I_S(x,R) = \begin{cases} 1 \text{ if } (x,R) \in S \\ 0 \text{ if } (x,R) \notin S \end{cases}$$

and

$$c(R) = 1/\int I_S(x,R)dx$$

as given by (1.4).

If $X \sim U\{||X-\theta||^2 \leq R^2\}$, X is the best invariant procedure with respect to quadratic loss (1.1) and is therefore a minimax estimator of θ. This follows from the results of Kiefer [14].

In this section, we will find classes of minimax estimators which are better than X when the loss is sum of squared errors (1.1) and $p \geq 3$.

2.1. Minimax estimators for dimension $p \geq 4$. Consider $X = [X_1, X_2, \ldots, X_p]'$, where X is one observation on a spherical uniform distribution with location parameter θ. When the loss is (1.1), we will prove that $\delta_a(X)$ given by

$$(2.1.1) \quad \delta_a(X) = \left(1 - (a/||X||^2)\right)X$$

is a minimax estimator of θ when $0 \leq a \leq 2c_o R^2$, where

$$(2.1.2)\quad c_o = \begin{cases} (p-2)/(p+2) & \text{when } p = 4 \\ .16 & \text{when } p = 3 \end{cases}.$$

When $p = 4$ the results we prove differ somewhat from those for $p = 3$. Hence, we consider only dimensions $p \geq 4$ in this section and dimension $p = 3$ will be dealt with in Section 2.2.

Clearly $\delta_a(X)$ will be minimax, with respect to quadratic loss (1.1), if the risk of $\delta_a(X)$, $R\left(\delta_a(X), \theta\right)$, dominates (is less than or equal to for all θ) the risk, $R(X,\theta)$, of the best invariant procedure X.

In order to prove this, we will show, for all θ,

$$\begin{aligned} &R(X,\theta) - R\left(\delta_a(X), \theta\right) \\ (2.1.3)\quad &= E_\theta\|X-\theta\|^2 - E_\theta\|(1 - a\|X\|^{-2})X - \theta\|^2 \\ &= aE_\theta[2 - 2(\theta'X)\|X\|^{-2} - a\|X\|^{-2}] \end{aligned}$$

is non-negative.

Many of the calculations required to obtain expressions for the difference in risks have been deferred to the Appendix in Section 6 to enable a smoother presentation of the proofs in this section.

It is straightforward to use (2.1.3) and Lemma 6.1.2, to obtain the following expressions for the difference in risks

$$\begin{aligned} &R(X,\theta) - R\left(\delta_a(X), \theta\right) \\ (2.1.4)\quad &= a[2 - 2E_{\|\theta\|}\left((\|\theta\|X_1)\|X\|^{-2}\right) - aE_{\|\theta\|}(\|X\|^{-2})] \\ &= a[2 - E_0 2\|\theta\|(X_1 + \|\theta\|)\left((X_1 + \|\theta\|)^2 + \|Y\|^2\right)^{-1} - aE_0\left((X_1+\|\theta\|)^2 + \|Y\|^2\right)^{-1}] \end{aligned}$$

where $\|Y\|^2 = \|X\|^2 - X_1^2 = \sum_{i=2}^{p} X_i^2$ and $E_{\|\theta\|}$ is the expected value when $\theta = [\|\theta\|, 0, 0, \ldots, 0]'$. We point out at this time, (2.1.4) indicates that the difference in risks only depends on $||\theta||$. Substituting in (2.1.4), the expressions for the expected values given by 6.1.6 and 6.1.7, we obtain the following:

$$\left(R(X,\theta) - R\left(\delta_a(X),\theta\right)\right)\Big/ aM = D(a,||\theta||)$$

$$(2.1.5) = \frac{4}{(p-1)} \int_0^R \frac{(R^2-y^2)^{\frac{p-1}{2}} [R^4 - 3||\theta||^2R^2 + 4||\theta||^2(R^2-y^2)]dy}{d_{R,||\theta||}(y)}$$

$$- \frac{2a}{(p-2)} \int_0^R \frac{(R^2-y^2)^{\frac{p-3}{2}} [R^4 - ||\theta||^2R^2 + 2||\theta||^2(R^2-y^2)]dy}{d_{R,||\theta||}(y)}$$

where

$$(2.1.6) \quad d_{R,||\theta||}(y) = (R^2-||\theta||^2)^2 + 4||\theta||^2(R^2-y^2)$$

and

$$(2.1.7) \quad M = \left[\frac{2}{p} R^2 \int_0^R (R^2-y^2)^{\frac{p-3}{2}} dy\right]^{-1} .$$

Clearly, $R(X,\theta) - R\left(\delta_a(X), \theta\right) \geq 0$ if $0 \leq a \leq b(||\theta||)$ where

$$(2.1.8) \quad b(||\theta||) = \frac{\left(2/(p-1)\right)\int_0^R \frac{(R^2-y^2)^{\frac{p-1}{2}} [R^4 - 3||\theta||^2R^2 + 4||\theta||^2(R^2-y^2)]\, dy}{d_{R,||\theta||}(y)}}{\left(1/(p-2)\right)\int_0^R \frac{(R^2-y^2)^{\frac{p-1}{2}} [R^4 - ||\theta||^2R^2 + 2||\theta||^2(R^2-y^2)]\, dy}{d_{R,||\theta||}(y)}} .$$

With respect to the density

$$(2.1.9) \quad g_{p,\|\theta\|}(y) = \begin{cases} \left((R^2-y^2)^{\frac{p-1}{2}} / d_{R,\|\theta\|}(y)\right) \Big/ \left(\int_0^R \left((R^2-y^2)^{\frac{p-1}{2}} / d_{R,\|\theta\|}(y)\right) dy\right) & \text{for } 0 \le y \le R \\ 0 \text{ elsewhere} \end{cases}$$

(this is the density given by (6.2.1) when $q = (p-1)/2$),

$$(2.1.10) \quad b(\|\theta\|) = \frac{2(p-2)}{(p-1)} \left[\frac{R^4 - 3\|\theta\|^2 R^2 + 4\|\theta\|^2 E_{\|\theta\|}(R^2-Y^2)}{(R^4-\|\theta\|^2R^2)\, E_{\|\theta\|}(R^2-Y^2)^{-1} + 2\|\theta\|^2} \right] .$$

Using (2.1.9), (2.1.10) and (6.1.1) we easily obtain the following:

$$(2.1.11) \quad b(0) = \left(2(p-2)/p\right)R^2$$

$$(2.1.12) \quad b(R) = \left(2(p-2)^2/p(p-1)\right)R^2$$

and

$$(2.1.13) \quad \lim_{\|\theta\|\to\infty} b(\|\theta\|) = \left(2(p-2)/(p+2)\right)R^2 .$$

Note that when $p \ge 4$, $b(0) > b(R) \ge \lim_{\|\theta\|\to\infty} b(\|\theta\|)$. It will be shown in Theorem 2.1.1 that $\lim_{\|\theta\|\to\infty} b(\|\theta\|)$, in fact, is less than or equal to $b(\|\theta\|)$ for all $\|\theta\|$. Note also, $\lim_{\|\theta\|\to\infty} b(\|\theta\|) = \left(2(p-2)/(p+2)\right)R^2 = \left(2p/(p+2)\right)\left(E_0(\|X\|^{-2})\right)$ as stated in Lemma 6.1.1.

Theorem 2.1.1: If X is one observation on a p-dimensional family of the form (1.4) then $\delta_a(X)$ given by (2.1.1) is minimax provided $0 \le a \le \left(2(p-2)/(p+2)\right)R^2$, $p \ge 4$ and the loss is sum of squared errors (1.1). Furthermore, $\delta_a(X)$ is actually better than X for $0 < a \le \left(2p/(p+2)\right)\left(1/E_0(\|X\|^{-2})\right)$.

Proof: Clearly by (2.1.4) and (2.1.5)

$R(X,\theta) - R\left(\delta_a(X), \theta\right) = aMD(a,\|\theta\|) \ge aMD\left(\left(2(p-2)/(p+2)\right)R^2, \|\theta\|\right)$ provided $0 \le a \le \left(2(p-2)/(p+2)\right)R^2 = \left(2p/(p+2)\right)\left(1/E_0(\|X\|^{-2})\right)$.

To prove $\delta_a(X)$ is minimax we will show $D\Big(\big(2(p-2)/(p+2)\big)R^2, \|\theta\|\Big) \geq 0$. In order to do this we consider the following 3 cases: $\|\theta\|^2 \leq R^2$, $R^2 < \|\theta\|^2 \leq (p/4)R^2$ and $(p/4)R^2 < \|\theta\|^2$.

Case 1: $\|\theta\|^2 \leq R^2$

With respect to the density given by (2.1.9)

$$
\begin{aligned}
& D_1\Big(\big(2(p-2)/(p+2)\big)R^2, \|\theta\|\Big) \\
(2.1.14)\quad &= D\Big(\big(2(p-2)/(p+2)\big)R^2, \|\theta\|\Big) \Big/ \int_0^R \Big((R^2-y^2)^{\frac{p-1}{2}} \big/ d_{R,\|\theta\|}(y)\Big)dy \\
&= \big(4/(p-1)\big)[R^4 - 3\|\theta\|^2R^2 + 4\|\theta\|^2 E_{\|\theta\|}(R^2-Y^2)] \\
&\quad - \big(4R^2/(p+2)\big)[(R^4-\|\theta\|^2R^2)E_{\|\theta\|}(R^2-Y^2)^{-1} + 2\|\theta\|^2]
\end{aligned}
$$

By Lemma 6.2.1, when $\|\theta\|^2 \leq R^2$, $g_{p,\|\theta\|}(y)$ has monotone likelihood ratio (MLR) non-decreasing in y. By Lehmann [15], page 74, this implies $E_{\|\theta\|}(R^2-Y^2) \geq E_R(R^2-Y^2) = \big((p-1)/p\big)R^2$ and $E_{\|\theta\|}(R^2-Y^2)^{-1} \leq E_R(R^2-Y^2)^{-1} = \big((p-2)/(p-3)\big)R^{-2}$. Therefore,

$$
\begin{aligned}
D_1\Big(\big(2(p-2)/(p+2)\big)R^2, \|\theta\|\Big) &\geq \big(4/(p-1)\big)[R^4 - 3\|\theta\|^2R^2 + 4\|\theta\|^2R^2\big((p-1)/p\big)] \\
&\quad - \big(4/(p+2)\big)R^2\,[(R^2-\|\theta\|^2)\big((p-2)/(p-3)\big) + 2\|\theta\|^2] \\
&= \Big(8(p-4)\big/\big(p(p-1)(p+2)(p-3)\big)\Big)R^2\,[pR^2 - 3\|\theta\|^2] \geq 0,
\end{aligned}
$$

when $p \geq 4$. Hence, $D\Big(\big(2(p-2)/(p+2)\big)R^2, \|\theta\|\Big) \geq 0$.

Case 2: $R^2 < \|\theta\|^2 \leq (p/4)R^2$

(Note that when $p = 4$, this case is vacuous.)

When $R^2 < \|\theta\|^2 \leq (p/4)R^2$, by Lemma 6.2.1, $g_{p,\|\theta\|}(y)$ given by (2.1.9) has MLR non-increasing in y. Therefore, $E_{\|\theta\|}(R^2-Y^2) \geq E_R(R^2-Y^2) = \big((p-1)/p\big)R^2$

and $E_{\|\theta\|}(R^2-Y^2)^{-1} \geq \lim_{\|\theta\|\to\infty} E_{\|\theta\|}(R^2-Y^2)^{-1} = \left(p/(p-1)R^2\right)$. Hence, for $\|\theta\|^2 \leq (p/4)R^2$,

$$\begin{aligned} D_1\left(\left(2(p-2)/(p+2)\right)R^2, \|\theta\|\right) &\geq \left(4/(p-1)\right)[R^4-3\|\theta\|^2R^2+4\left((p-1)/p\right)\|\theta\|^2R^2] \\ &\quad - \left(4R^2/(p+2)\right)[\left(p/(p-1)\right)(R^2-\|\theta\|) + 2\|\theta\|^2] \\ &= \left(8R^2/\left(p(p-1)(p+2)\right)\right)[pR^2-4\|\theta\|^2] \geq 0. \end{aligned}$$

Clearly, by (2.1.14), $D\left(\left(2(p-2)/(p+2)\right)R^2, \|\theta\|\right) \geq 0$.

Case 3: $\|\theta\|^2 \geq (p/4)R^2$

In order to show that $D\left(\left(2(p-2)/(p+2)\right)R^2,\|\theta\|\right) \geq 0$, when $\|\theta\|^2 > (p/4)R^2$, we first obtain a new expression for $D\left(\left(2(p-2)/(p+2)\right)R^2,\|\theta\|\right)$.

By applying (6.1.3) to (2.1.5), simple calculations lead to

$$\begin{aligned} &D\left(\left(2(p-2)/(p+2)\right)R^2, \|\theta\|\right) \\ &= \left(1/\left(p(p-1)(p+2)\right)\right)\Big[[(p^2+4p-8)R^2-p(p+2)\|\theta\|^2]\int_0^R(R^2-y^2)^{\frac{p-3}{2}}dy \\ &\quad + (\|\theta\|^4-R^4)p[(p-4)R^2+(p+2)\|\theta\|^2]\int_0^R\left((R^2-y^2)^{\frac{p-3}{2}}/d_{R,\|\theta\|}(y)\right)dy\Big] \end{aligned}$$

Hence, clearly by Lemma 6.1.5,

$$\begin{aligned} &D\left(\left(2(p-2)/(p+2)\right)R^2, \|\theta\|\right) = \\ &\left(\int_0^R(R^2-y^2)^{\frac{p-3}{2}}dy/\left(p(p-1)(p+2)\right)\right)\Big[(p^2+4p-8)R^2 - p(p+2)\|\theta\|^2 \\ &\quad + (\|\theta\|^4-R^4)\, p[(p-4)R^2 + (p+2)\|\theta\|^2]\,[h(\|\theta\|,R)]_p\Big]. \\ &= \left(\int_0^R(R^2-y^2)^{\frac{p-3}{2}}dy/\left(p(p-1)(p+2)a_0\|\theta\|^2\right)\right)D_2\left(\left(2(p-2)/(p+2)\right)R^2, \|\theta\|\right), \end{aligned}$$

where $[h(\|\theta\|,R)]_p = \left(1/\left(a_0\|\theta\|^2(R^2+\|\theta\|^2)\right)\right)\sum_{i=0}^{\infty}(-1)^i a_i(R^2/\|\theta\|^2)^i$

and $a_i = [\left(p-2(i+1)\right)/\left(p+2(i-1)\right)]a_{i-1}$ for $i = 0,1,\ldots$.

Therefore,

$$D_2\left(\left(2(p-2)/(p+2)\right)R^2,\ \|\theta\|\right) =$$

$$a_0\|\theta\|^2[-p(p+2) + p(p+2)] + \|\theta\|^2R^2[(p^2+4p-8)a_0 - p(p+2)a_1-6pa_0]$$

$$+ R^4[p(p+2)a_2 + 6pa_1 + p(-p+4)a_0] + p(p-4)\sum_{i=1}^{\infty}(-1)^i a_i R^{2i+4}\|\theta\|^{-2i}$$

$$+ 6p\sum_{i=2}^{\infty}(-1)^{i-1}a_i R^{2i+2}\|\theta\|^{-2i+2} + p(p+2)\sum_{i=3}^{\infty}(-1)^i a_i R^{2i}\|\theta\|^{-2i+4}$$

$$= (pR^6/\|\theta\|^2)\sum_{i=0}^{\infty}(-1)^i c_i(R^2/\|\theta\|^2)^i$$

where

$$c_i = (p-4)a_{i+1} - 6a_{i+2} - (p+2)a_{i+3}$$

$$= [8(i+1)(p^2-3p+3)/\left(\left(p+2(i+2)\right)\left(p+2(i+1)\right)\right)]a_{i+1} \qquad i = 0,1,2,\ldots.$$

Since $\|\theta\|^2 > (p/4)R^2$,

$$\sum_{i=0}^{\infty}(-1)^i c_i(R^2/\|\theta\|^2)^i = \sum_{i=0}^{\infty}c_{2i}(R^2/\|\theta\|^2)^{2i} - \sum_{i=0}^{\infty}c_{2i+1}(R^2/\|\theta\|^2)^{2i+1}$$

$$\geq \sum_{i=0}^{\infty}c_{2i}(R^2/\|\theta\|^2)^{2i} - (4/p)\sum_{i=0}^{\infty}c_{2i+1}(R^2/\|\theta\|^2)^{2i}.$$

Clearly, $D\left(\left(2(p-2)/(p+2)\right)R^2,\ \|\theta\|\right) \geq 0$, if $c_{2i+1} \leq (p/4)c_{2i}$. Note, $(2i+1)p^2 + 2(4i^2+4i-1)p + 16(2i^2+15i+3) \geq p^2 - 2p + 48 > 0$, implying $c_{2i+1} = [(2i+2)\left(p-2(2i+3)\right)/\left((2i+1)\left(p+2(2i+3)\right)\right)]c_{2i} \leq (p/4)c_{2i}$.

By combining these 3 cases, we have now proven that $R(X,\theta) - R\left(\delta_a(X),\theta\right) \geq 0$ for all θ when $0 \leq a \leq \left(2(p-2)/(p+2)\right)R^2$ and so $\delta_a(X)$ is at least as good as X for these a's. However, when $0 < a \leq \left(2(p-2)/(p+2)\right)R^2$

$$
\begin{aligned}
R(X,0) &- R\left(\delta_a(X),0\right) \\
&= a[2 - aE_0(\|X\|^{-2})] \text{ by } (2.1.4) \\
&= a[2 - \left(ap/\left((p-2)R^2\right)\right)] \text{ by Lemma 6.1.1} \\
&\geq a[2 - 2p/(p+2)\ ] \\
&= 4a/(p+2) > 0
\end{aligned}
$$

implying $\delta_a(X)$ is better than X. This completes the proof of Theorem 2.1.1.

2.2. Minimax estimators for p = 3. We saw in Section 2.1 that with respect to quadratic loss (1.1), $R(X,\theta) - R\left(\delta_a(X),\theta\right)$ is non-negative, provided $0 \leq a \leq b(\|\theta\|)$ where $b(\|\theta\|)$ is defined by (2.1.8). For $p = 3$, by (2.1.11) - (2.1.13), $b(0) = (2/3)R^2$, $b(R) = (1/3)R^2$ and $\lim_{\|\theta\|\to\infty} b(\|\theta\|) = (2/5)R^2$. Hence, the best possible result would be to show $\delta_a(X)$ is minimax for $0 \leq a \leq (R^2/3)$. It will become clear that this is not true.

Clearly, by (2.1.5) and (2.1.6), when $p = 3$

$$
\begin{aligned}
D(a,\|\theta\|) &= 2\int_0^R \left((R^2-y^2)/d_{R,\|\theta\|}(y)\right)[R^4-3\|\theta\|^2R^2+4\|\theta\|^2(R^2-y^2)]dy \\
&\quad - 2a\int_0^R \left(1/d_{R,\|\theta\|}(y)\right)[R^4 - \|\theta\|^2R^2 + 2\|\theta\|^2(R^2-y^2)]dy.
\end{aligned}
\tag{2.2.1}
$$

However, applying (6.1.3) and

$$\int_0^R \left(1/d_{R,\|\theta\|}(y)\right)dy = \left(4\|\theta\|(R^2+\|\theta\|^2)\right)^{-1}\log\left((R+\|\theta\|)/(R-\|\theta\|)\right)^2$$ to (2.2.1)

we get

$$
\begin{aligned}
D(a,\|\theta\|) &= \\
&(R/3)\left[R^2+(3/2)(R^2-\|\theta\|^2-2a)\ [1 + \left((R^2-\|\theta\|^2)/4\|\theta\|R\right)\log\left((R+\|\theta\|)/(R-\|\theta\|)\right)^2]\right]
\end{aligned}
\tag{2.2.2}
$$

Therefore, when $\|\theta\| \geq R$

$$D(R^2/3,\ \|\theta\|) = (1/3)\Big[R^2+(R^2/2)(R^2-3\|\theta\|^2)[1+\big((R^2-\|\theta\|^2)/\|\theta\|\ R\big)\sum_{n=1}^{\infty}\frac{(R/\|\theta\|)^{2n-1}}{2n-1}]\Big]$$

$$= \big((\|\theta\|^2-R^2)/6\|\theta\|\big)[-3R\|\theta\| + (3\|\theta\|^2-R^2)\sum_{n=1}^{\infty}\frac{(R/\|\theta\|)^{2n-1}}{2n-1}]$$

$$= \big(4R^2(\|\theta\|^2-R^2)/6\|\theta\|\big)\sum_{n=1}^{\infty}\Big(\frac{n-1}{(2n+1)(2n-1)}\Big)(R/\|\theta\|)^{2n-1} \geq 0.$$

When $\|\theta\| \leq R$,

$$D(R^2/3,\ \|\theta\|) = \big((R^2-\|\theta\|^2)/6\|\theta\|\big)[3R\|\theta\| + (R^2-3\|\theta\|^2)\sum_{n=1}^{\infty}\frac{(\|\theta\|/R)^{2n-1}}{2n-1}]$$

$$= \big(2(R^2-\|\theta\|^2)R/6\big)[1-\sum_{n=1}^{\infty}\Big(\frac{n+1}{(2n+1)(2n-1)}\Big)(\|\theta\|/R)^{2n}]$$

$$= \big(2(R^2-\|\theta\|^2)R/6\big)\ D_3(R^2/3,\ \|\theta\|)$$

When $K \geq 4$, $\sum_{n=1}^{k}\frac{n+1}{(2n+1)(2n-1)} \geq 1$, indicating that $D_3(R^2/3,\ \|\theta\|) \leq 0$, when $R^2 - \epsilon \leq \|\theta\|^2 \leq R^2$, for some $\epsilon > 0$. Clearly, this implies $D(R^2/3,\ \|\theta\|) \leq 0$ for the same $\|\theta\|$'s. Hence, since there exists a $\|\theta\|$ such that $R(X,\theta) - R\big(\delta_a(X),\theta\big) = aMD(a,\|\theta\|) \leq 0$ when $a = R^2/3$, $\delta_a(X)$ is not minimax for all a satisfying $0 \leq a \leq R^2/3$.

By a simple inequality, $D_3(R^2/3,\|\theta\|) = 1 - \sum_{n=1}^{\infty}\Big(\frac{n+1}{(2n+1)(2n-1)}\Big)(\|\theta\|/R)^{2n}$

$$\geq 1 - (2\|\theta\|^2/3R^2) - (1/5)\ [1 - (\|\theta\|^2/R^2) + \big(R^2/(R^2-\|\theta\|^2)\big)]$$

$$\geq 0 \text{ when } 0 \leq \|\theta\| \leq \sqrt{(25-\sqrt{205})/14}\ R.$$

Since $\sqrt{(25-\sqrt{205})/14} > .85$, we conclude that

$$D(R^2/3,\ \|\theta\|) \geq 0 \text{ when } 0 \leq \|\theta\| \leq .85R \text{ and } \|\theta\| \geq R. \tag{2.2.3}$$

For several values of $\|\theta\|$ between .85R and R, we calculate $b(\|\theta\|)$ given by (2.1.8) when $R = 1$. Our findings are summarized in the following table:

Table 2.2.1

Evaluation of $b(\|\theta\|)$
(given by (2.1.8) when R=1)

$\|\theta\|$	$b(\|\theta\|)$
.85	.3751
.90	.3493
.95	.3293
.96	.3268
.97	.3252
.98	.3248
.99	.3264
1.00	.3333

From Table 2.2.1 we see that $b(\|\theta\|)$ is in fact $< 1/3$ when the $\|\theta\|$ is close to 1.

In Theorem 2.2.2, we will prove $\delta_a(X)$ is minimax for $0 \leq a \leq (.75)/E_0(\|X\|^{-2}) = R^2/4$. However, since the smallest value of $b(\|\theta\|)$ we compute is .3248, it seems almost certain that $\delta_a(X)$ is minimax for a larger class of a's, namely, $\delta_a(X)$ is minimax for $0 \leq a \leq .32R^2 = (.96)/E_0(\|X\|^{-2})$. This is stated in Theorem 2.2.1.

In addition, since the proofs in the following sections which use the minimaxity of $\delta_a(X)$ for a larger class of a's are certainly true for the smaller class, Theorem 2.2.1 will be cited over Theorem 2.2.2.

Theorem 2.2.1: If X is a random vector with a 3-dimensional spherical uniform distribution then $\delta_a(X)$ given by (2.1.1) is minimax provided $0 \leq a \leq (.96)/E_0(\|X\|^{-2}) = .32R^2$ and the loss is quadratic loss.

Theorem 2.2.2: If X has a 3-dimensional spherical uniform distribution then $\delta_a(X)$ given by (2.2.1) is better than X provided $0 < a \leq R^2/4 = (.75)/E_0(\|X\|^{-2})$ and the loss is given by (1.1).

Proof: When $0 \leq \|\theta\| \leq .85R$ and $\|\theta\| \geq R$, we have already shown, by (2.2.3), $R(X,\theta) - R\left(\delta_a(X),\theta\right) = aMD(a,\|\theta\|) \geq aMD(R^2/4,\|\theta\|) \geq aMD(R^2/3,\|\theta\|) \geq 0.$

We now show $D(R^2/4,\|\theta\|) \geq 0$ when $.85R \leq \|\theta\| \leq R$.

From (2.2.2),

$$D(R^2/4,\|\theta\|) = (R/3)\left[R^2 + (3/4)(R^2-2\|\theta\|^2)[1 + \left((R^2-\|\theta\|^2)/4\|\theta\|R\right)\log\left((R+\|\theta\|)/(R-\|\theta\|)\right)^2\right]$$

$$= (R/3)\left[R^2 + \left(3(R^2-\|\theta\|^2)/\|\theta\|^2\right)[\|\theta\|^2 + (R^2-\|\theta\|^2)\sum_{n=1}^{\infty}\frac{(\|\theta\|/R)^{2n}}{2n-1}]\right].$$

Note,

$$\|\theta\|^2 + (R^2-\|\theta\|^2)\sum_{n=1}^{\infty}\frac{(\|\theta\|/R)^{2n}}{2n-1}$$

$$\leq (2R^2-\|\theta\|^2)(\|\theta\|^2/R^2) + \left((R^2-\|\theta\|^2)/3\right)[-1 -(\|\theta\|^2/R^2) + \left(R^2/(R^2-\|\theta\|^2)\right)]$$

$$= (2\|\theta\|^2/3R^2)(3R^2-\|\theta\|^2).$$

Therefore,

$$D(R^2/4,\|\theta\|) \geq (R/3)[R^2 + (1/2R^2)(R^2-2\|\theta\|^2)(3R^2-\|\theta\|^2)]$$

$$= (1/6R)[5R^4 - 7\|\theta\|^2R^2 + 2\|\theta\|^4] \geq 0.$$

Hence, we have proven that $R\left(\delta_a(X),\theta\right) \leq R(X,\theta)$ for all θ when $0 \leq a \leq (R^2/4)$. To show $\delta_a(X)$ is actually better than X when $0 < a \leq (R^2/4)$, we prove the risk of X at $\theta = 0$ is strictly greater than the risk of $\delta_a(X)$ at $\theta = 0$.

$$R(X,0) - R(\delta_a(X),0)$$

$$= a[2 - (3a/R^2)]$$

$$\geq a[2 - (3/4)]$$

$$= (5/4)a > 0.$$

This completes the proof.

Throughout this paper we will refer to the result of Theorem 2.2.1 when $p = 3$.

2.3. A larger class of minimax estimators when $p \geq 3$. We now consider a new class of estimators for the location parameter θ when X has a p-dimensional spherical uniform distribution. The estimators are of the form considered by Baranchik [2] and are given by

$$(2.3.1) \quad \delta_{a,r}(X) = \left(1 - a\left(r(\|X\|^2)/\|X\|^2\right)\right)X.$$

If $r(\|X\|^2)$ is non-decreasing, $0 \leq r(\cdot) \leq 1$, and $0 \leq a \leq (2b_o)/E_0(\|X\|^{-2})$ where

$$(2.3.2) \quad b_o = \begin{cases} p/(p+2) & \text{when } p \geq 4 \\ .48 & \text{when } p = 3 \end{cases}$$

then we will prove $\delta_{a,r}(X)$ given by (2.3.1) is minimax.

When $r(\cdot) \equiv 1$, this result coincides with those given in Theorems 2.1.1 and 2.2.1. Hence, we have a larger class of minimax estimators.

Theorem 2.3.1. If $X = [X_1, X_2, \ldots, X_p]'$ has a p-dimensional spherical uniform distribution about θ, then the risk of $\delta_{a,r}(X)$, where $\delta_{a,r}(X)$ is defined by (2.3.1), dominates (is less than or equal to) the risk of X with respect to quadratic loss (1.1), provided $r(\|X\|^2)$ is non-decreasing, and $0 \leq a \leq (2b_o)/E_0(\|X\|^{-2})$, where b_o is defined by (2.3.2).

Proof: Note that $(2b_o)/E_0(\|X\|^{-2}) = 2c_oR^2$ when $p \geq 3$ where $c_o = (p-2)/(p+2)$ when $p \geq 4$ and $c_o = .16$ when $p = 3$ as given by (2.1.2). Since $0 \leq r(\cdot) \leq 1$ and $o \leq a \leq (2b_o)/E_0(\|X\|^{-2}) = 2c_oR^2$,

$$
\begin{aligned}
R(X,\theta) &- R\left(\delta_{a,r}(X),\theta\right) \\
&= aE_\theta[r(\|X\|^2)\left(2X'(X-\theta)\right)\|X\|^{-2} - ar^2(\|X\|^2)\|X\|^{-2}] \\
&\geq 2aE_\theta\left[r(\|X\|^2)[X'(X-\theta)\|X\|^{-2} - c_oR^2\|X\|^{-2}]\right] \\
&= 2aE_{\|\theta\|}\left[r(\|X\|^2)[1 - \|\theta\|X_1\|X\|^{-2} - c_oR^2\|X\|^{-2}]\right].
\end{aligned}
$$

Note that the difference in risks depends only on $\theta = [\|\theta\|, 0, 0, \ldots, 0]'$. We will show

$$E_{\|\theta\|}\left[r(\|X\|^2)[1 - \|\theta\|X_1\|X\|^{-2} - c_oR^2\|X\|^{-2}]\right] \geq 0.$$

We first consider $\|\theta\|^2 \geq (1 - 2c_o)R^2$.

Case 1: $\|\theta\|^2 \geq (1 - 2c_o)R^2$

Lemma 6.2.2. states for each fixed $\|\theta\|$ satisfying $\|\theta\|^2 \geq (1 - 2c_o)R^2$, $E_{\|\theta\|}[(\|\theta\|X_1 + c_oR^2)\|X\|^{-2}\big|\|X\|^2]$ is non-increasing in $\|X\|^2$. Since $r(\|X\|^2)$ is a non-decreasing function,

$$
\begin{aligned}
E_{\|\theta\|}&\left[r(\|X\|^2)[1 - (\|\theta\|X_1 + c_oR^2)\|X\|^{-2}]\right] \\
&= E_{\|\theta\|}\left[r(\|X\|^2)[1 - E_{\|\theta\|}\left((\|\theta\|X_1+c_oR^2)\|X\|^{-2}\big|\|X\|^2\right)]\right] \\
&\geq \left(E_{\|\theta\|}\ r(\|X\|^2)\right)[\left(R(X,\theta) - R\left(\delta_{2c_oR^2}(X),\theta\right)\right)/4c_oR^2].
\end{aligned}
$$

This is non-negative for all θ since $\delta_{2c_oR^2}$ is better than X (Theorems 2.1.1 and 2.2.1).

Since $1 - 2c_o = (6-p)/(p+2)$ when $p \geq 4$ and $1 - 2c_o = .68$ when $p = 3$, the proof of the theorem is complete for all $\|\theta\|$ when $p \geq 6$, for $\|\theta\|^2 \geq \big((6-p)/(p+2)\big)R^2$ when $p = 4, 5$ and when $\|\theta\|^2 \geq .68R^2$ when $p = 3$.

Case 2: $\|\theta\|^2 \leq .68R^2$ and $p = 3$

If $\|X\|^2 = Z$, we see by Lemma 6.1.6, when $p = 3$, the joint density of X_1 and Z, $f_{\|\theta\|}(x_1,z)$, is given by

$$f_{\|\theta\|}(x_1,z) = (3/4R^3)I_{S_1 \cup S_2}(x_1,z)$$

where

$$S_1 = \Big((x_1,z): \ \big((z-R^2+\|\theta\|^2)/2\|\theta\|\big) \leq x_1 \leq \sqrt{z} \ , \ (R-\|\theta\|)^2 \leq z \leq (R+\|\theta\|)^2\Big)$$

and

$$S_2 = \Big((x_1,z): \ -\sqrt{z} \leq x_1 \leq \sqrt{z}, \ 0 \leq z \leq (R+\|\theta\|)^2\Big).$$

Hence,

$$E_{\|\theta\|}r(\|X\|^2)[1 - (\|\theta\|X_1 + c_oR^2)\|X\|^{-2}]$$
$$= (3/4R^3)\int_0^{(R-\|\theta\|)^2}\big(2r(z)/\sqrt{z}\big)(z-.16R^2)dz$$
$$+ (3/4R^3)\int_{(R-\|\theta\|)^2}^{(R+\|\theta\|)^2}\big(r(z)/4z\big)\Big(\sqrt{z} - \big((z-R^2+\|\theta\|^2)/2\|\theta\|\big)\Big)(3z-2\|\theta\|\sqrt{z}+.36R^2-\|\theta\|^2)dz$$

If

$$h(z) = \begin{cases} z - .16R^2 & \text{when } 0 \leq \sqrt{z} \leq R-\|\theta\| \\ 3z - 2\|\theta\|\sqrt{z} + (.36R^2-\|\theta\|^2) & \text{when } R-\|\theta\| \leq \sqrt{z} \leq R+\|\theta\| \end{cases}$$

then for $0 \leq \|\theta\| \leq .6R$,

$$h(z) \leq 0 \quad \text{when } 0 \leq \sqrt{z} \leq .4R$$
$$\geq 0 \quad \text{when } .4R \leq \sqrt{z} \leq R+\|\theta\|.$$

Since $r(z)$ is non-decreasing, $r(z)h(z) \geq r(.16R^2)\ h(z)$.

For $\|\theta\| \geq .6R$,

$$h(z) \leq 0 \text{ when } 0 \leq \sqrt{z} \leq (\|\theta\|/3) + (2\sqrt{\|\theta\|^2-.27R^2}\ /3)$$

$$\geq 0 \text{ when } (\|\theta\|/3) + (2\sqrt{\|\theta\|^2-.27R^2}/3) \leq \sqrt{z} \leq R + \|\theta\|$$

and so

$$r(z)h(z) \geq r\left(\left((\|\theta\|/3) + (2\sqrt{\|\theta\|^2-.27R^2}\ /3)\right)^2\right)h(z).$$

Hence, in either case, there exists a z_o such that

$$E_{\|\theta\|}r(\|X\|^2)[1 - (\|\theta\|X_1+c_oR^2)\|X\|^{-2}]$$

$$= E_{\|\theta\|}r(z)h(z)$$

$$\geq r(z_o)E_{\|\theta\|}h(z)$$

$$\geq r(z_o)E_{\|\theta\|}[1 - (\|\theta\|X_1+c_oR^2)\|X\|^{-2}]$$

$$= \left(R(X,\theta) - R\left(\delta_{2c_oR^2}(X),\theta\right)\right)/4c_oR^2$$

≥ 0 by Theorem 2.2.1.

The proof is complete for this case.

Case 3: $0 \leq \|\theta\| \leq \left(2(p-1)/p(p+2)\right)R$ and $p = 4, 5$.

Since $X_1 \leq \|X\|$ and $r(\|X\|^2)$ is non-decreasing,

$$E_{\|\theta\|}r(\|X\|^2)[1 - \|\theta\|X_1\|X\|^{-2} - c_oR^2\|X\|^{-2}]$$

$$\geq E_{\|\theta\|}r(\|X\|^2)[1 - E_{\|\theta\|}(\|\theta\|\ \|X\|^{-1} + c_oR^2\|X\|^{-2})].$$

Since $\|X\|$, by Lemma 6.2.3, is stochastically ordered in $\|\theta\|$,

$$E_{\|\theta\|}[\|\theta\|\ \|X\|^{-1} + c_oR^2\|X\|^{-2}]$$

$$\leq E_0[\|\theta\|\ \|X\|^{-1} + c_oR^2\|X\|^{-2}]$$

$$= \left(p/(p-1)\right)(\|\theta\|/R) + \left(p/(p+2)\right)$$

$$\leq 1 \quad \text{when} \quad \|\theta\| \leq \left(2(p-1)/p(p+2)\right)R.$$

Hence, $E_{||\theta||} r(||X||^2)[1 - ||\theta||X_1||X||^{-2} - c_o R^2||X||^{-2}] \geq 0$.

Hence, this implies the remaining cases are $(R^2/16) \leq ||\theta||^2 \leq (R^2/3)$ when $p = 4$. and $\left(64R^2/((49)(25))\right) \leq ||\theta||^2 \leq (R^2/7)$ when $p = 5$.

Case 4: $(R^2/16) \leq ||\theta||^2 \leq (R^2/3)$ and $p = 4$.

Utilizing (6.1.13), Lemma 6.1.5 and (2.1.5), for $p = 4$, simple calculations imply if

$$A(b) = E_{||\theta||}(2 - 2||\theta||X_1||X||^{-2} - bR^2||X||^{-2}) \text{ then}$$

$3R^4A(b) = 6(1-b)R^4 + 3(b-2)||\theta||^2R^2 + 2||\theta||4$. For fixed $b < 1$, there exists an a_o such that $A(b) \geq 0$ when $0 \leq ||\theta||^2 \leq a_oR^2$. Lemma 6.2.2 states that $E_{||\theta||}\left(\left(||\theta||X_1 + (b/2)R^2\right)||X||^{-2}\Big|\,||X||^2\right)$ is non-increasing in $||X||$ for $||\theta||^2 \geq (1-b)R^2$. Hence, when $b \geq 2c_o = 2/3$ and $a_oR^2 \geq ||\theta||^2 \geq (1-b)R^2$,

$$\begin{aligned} &E_{||\theta||} r(||X||^2)[2 - 2(||\theta||X_1 + c_oR^2)||X||^{-2}] \\ &\geq E_{||\theta||} r(||X||^2)[2 - 2\left(||\theta||X_1 + (b/2)R^2\right)||X||^{-2}] \\ &\geq E_{||\theta||} r(||X||^2)\ A(b) \geq 0. \end{aligned}$$

Calculations show for $b = 15/16$, $a_o > 1/8$. Therefore, for $(R^2/16) \leq ||\theta||^2 \leq (R^2/8)$, $E_{||\theta||} r(||X||^2)[2 - 2(||\theta||X_1 + c_oR^2)||X||^{-2}] \geq 0$.

Similarly, since for $b = 7/8$, $a_o > 1/4$ and for $b = 3/4$, $a_o > 1/3$, then

$$E_{||\theta||} r(||X||^2)[2 - 2(||\theta||X_1 + c_oR^2)||X||^{-2}] \geq 0$$

when $(R^2/8) \leq ||\theta||^2 \leq (R^2/4)$ and $(R^2/4) \leq ||\theta||^2 \leq (R^2/3)$, implying the desired conclusion for this case.

Case 5: $\left(64R^2/((25)(49))\right) \leq ||\theta||^2 \leq (R^2/7)$ and $p = 5$

As we did in the previous case, when $b \geq 2c_o = 6/7$ and $||\theta||^2 \geq (1-b)R^2$,

$$E_{\|\theta\|}r(\|X\|^2)[2 - 2(\|\theta\|X_1 + c_o R^2)\|X\|^{-2}]$$

$$\geq E_{\|\theta\|}r(\|X\|^2)[2 - 2(\|\theta\|X_1 + (b/2)R^2)\|X\|^{-2}]$$

$$\geq E_{\|\theta\|}r(\|X\|^2)\ A(b).$$

With respect to the density $g_{p,\|\theta\|}(y)$ for $p = 5$ given by (2.1.9), we have using (2.1.5)

$$A(b)/\left(M\int_0^R (R^2-y^2)^2 dy\right) = A^*(b)$$

$$= (R^4 - 3\|\theta\|^2R^2) + 4\|\theta\|^2 E_{\|\theta\|}(R^2-y^2)$$

$$- (2bR^2/3)[(R^4-\|\theta\|^2R^2)E_{\|\theta\|}(R^2-y^2)^{-1} + 2\|\theta\|^2]$$

where $M = [(2/5)\int_0^R (R^2-y^2)dy]^{-1}$ is given by (2.1.7). By Lemma 6.2.1, $g_{p,\|\theta\|}(y)$ has monotone likelihood ratio non-decreasing in y and hence, since $\|\theta\|^2 \leq R^2/7 \leq R^2$, when $p = 5$,

$$E_{\|\theta\|}(R^2-Y^2) \geq E_R(R^2-Y^2) = (4/5)R^2 \text{ and}$$

$$E_{\|\theta\|}(R^2-Y^2)^{-1} \leq E_R(R^2-Y^2)^{-1} = (3/2R^2),$$

implying

$$A^*(b) = (5R^4 + \|\theta\|^2R^2)/5 - (bR^2/3)(3R^2+\|\theta\|^2).$$

When $b = 24/25$, $A^*(24/25) \geq 0$ for $\|\theta\|^2 \leq R^2/3$. Hence, when $R^2/25 \leq \|\theta\|^2 \leq R^2/3$,

$$E_{\|\theta\|}[2 - 2(\|\theta\|X_1 + c_o R^2)\|X\|^{-2}]$$

$$\geq E_{\|\theta\|}r(\|X\|^2)A(24/25)$$

$$= E_{\|\theta\|}r(\|X\|^2)[M\int_0^R (R^2-y^2)^2 dy]\ A^*(24/25)$$

$$\geq 0.$$

The interval, $R^2/25 \leq \|\theta\|^2 \leq R^2/3$ includes the interval $\left(64R^2/((25)(49))\right) \leq \|\theta\|^2 \leq R^2/7$, thus the proof is complete. Q. E. D.

3. Minimax estimators for the mean vector of a p-dimensional spherically symmetric unimodal distribution with respect to quadratic loss.

3.1. A characterization of a spherically symmetric unimodal distribution.

Definition 3.1.1: A random vector X is said to have a p-dimensional spherically symmetric unimodal (s.s.u.) distribution about θ if the density of X with respect to Lebesgue measure is a non-increasing function of $\|X-\theta\|$.

In this section we will give necessary and sufficient conditions for a random vector to have a s.s.u. distribution about θ in accordance with definition 3.1.1.

Theorem 3.1.1: If X is a p x 1 random vector ($p \geq 1$) with a density $g(\|x-\theta\|)$ with respect to Lebesgue measure, then $g(\cdot)$ is a non-increasing function of $\|x-\theta\|$ if and only if

$$g(\|x-\theta\|) = \int c(R)\ I_S(x,R)dF(R)$$

where $I_S(x,R)$ is given in (1.4) and c(R) given in (1.4) equals c/R^p, where c is a positive constant and $F(\cdot)$ is a cdf on $(0, \infty)$.

Proof: Simply by (1.4), if

$$g(\|x-\theta\|) = \int c(R)I_S(x,R)dF(R) = \int_{\|x-\theta\|}^{\infty} c(R)dF(R)$$

then $g(\|x-\theta\|)$ is a non-increasing function of $\|x-\theta\|$.

Conversely, suppose $g(\cdot)$ is a non-increasing function of $\|x-\theta\|$.

We show in Lemma 6.1.1,

$$\begin{aligned} H(R) &= P(\|X-\theta\| \leq R) \\ &= \int_{(\|x-\theta\| \leq R)} g(\|x-\theta\|)dx \\ &= (M_o/c(R)) \int_0^R r^{p-1} g(r)dr , \end{aligned}$$

where $M_o = P/R^P$. Hence, since $c(R) = c/R^P$, $M_o/c(R) = P/c$, and therefore,

$$(3.1.1) \quad H(R) = (P/c) \int_0^R r^{p-1} g(r)dr \text{ .}$$

Consider the function, $F(R)$, given by

$$(3.1.2) \quad F(R) = H(R) - (R^P/c)g(R) \text{ .}$$

It follows from (3.1.1) that

$$(3.1.3) \quad \begin{aligned} F(R) &= (P/c) \int_0^R r^{p-1} g(r)dr - (R^P/c)g(R) \\ &= (P/c) \int_0^R y^{p-1}[g(Y) - g(R)]dy \text{ .} \end{aligned}$$

We will show that $F(R)$, given by (3.1.2), is a cdf and characterizes the density $g(||x-\theta||)$ i.e. $g(||x-\theta||) = \int c(R) I_S(x,R)dF(R)$.

Since $g(\cdot)$ is non-increasing , $g(y) - g(R) \geq 0$ for $0 \leq y \leq R$ and, if $R_1 \leq R_2$, then $g(y) - g(R_1) \leq g(y) - g(R_2)$. Hence, using the expression for $F(R)$ given by (3.1.3), we have, for $R_1 \leq R_2$,

$$\begin{aligned} F(R_1) &= (P/c)\int_0^{R_1} y^{p-1}[g(y) - g(R_1)]dy \\ &\leq (P/c)\int_0^{R_1} y^{p-1}[g(y) - g(R_2)]dy \\ &\leq (P/c)\int_0^{R_2} y^{p-2}[g(y) - g(R_2)]dy \\ &= F(R_2), \end{aligned}$$

thus implying $F(R)$ is non-decreasing.

Since $H(R)$, given by (3.1.1), is a cdf, $\lim_{R\to 0} H(R) = 0$ and $\lim_{R\to\infty} H(R) = 1$. Furthermore, for any $\varepsilon > 0$

$$2^p[H(\epsilon) - H(\epsilon/2)]$$

$$= 2^p(P/c) \int_{\epsilon/2}^{\epsilon} r^{p-1} g(r)dr$$

$$\geq 2^p \; p\left(g(\epsilon)/c\right) \int_{\epsilon/2}^{\epsilon} r^{p-1}dr$$

$$= \left(g(\epsilon)/c\right)[2^p \epsilon^p - \epsilon^p]$$

$$\geq g(\epsilon)(\epsilon^p/c) \; .$$

Hence, $0 = \lim\limits_{\epsilon \to 0} 2^p[H(\epsilon) - H(\epsilon/2] \geq \lim\limits_{\epsilon \to 0} g(\epsilon)(\epsilon^p/c) \geq 0$

and $\quad 0 = \lim\limits_{\epsilon \to \infty} 2^p[H(\epsilon) - H(\epsilon/2] \geq \lim\limits_{\epsilon \to \infty} g(\epsilon)(\epsilon^p/c) \geq 0$.

Thus,

$$\lim_{R \to 0} F(R) = \lim_{R \to 0} [H(R) - g(R)(R^p/c)] = 0$$

and

$$\lim_{R \to \infty} F(R) = \lim_{R \to \infty} [H(R) - g(R)(R^p/c)] = 1 \; .$$

Therefore, $F(R)$ is a cdf on $(0, \infty)$.

Furthermore, an integration by parts and (3.1.2) yields the following:

$$\int_{||x-\theta||}^{\infty} (c/R^p)dF(R) = (c/R^p)F(R)\Big/_{||x-\theta||}^{\infty} - c\int_{||x-\theta||}^{\infty} F(R)d(1/R^p)$$

$$= -\left(cF(||x-\theta||)/||x-\theta||^p\right) + cp \int_{||x-\theta||}^{\infty} \left(F(R)/R^{p+1}\right)dR$$

(3.1.4)

$$= g(||x-\theta||) - \left(cH(||x-\theta||)/||x-\theta||^p\right)$$

$$- p \int_{||x-\theta||}^{\infty} \left(g(R)/R\right)dR + cp \int_{||x-\theta||}^{\infty} \left(H(R)/R^{p+1}\right)dR \; .$$

(See Fisz [9], page 187, for example, for the integration by parts used above).

Substituting expression (3.1.1) for H(R) and applying Fubini's Theorem,

$$cp \int_{\|x-\theta\|}^{\infty} \left(H(R)/R^{p+1}\right) dR = p^2 \int_{\|x-\theta\|}^{\infty} \int_0^R \left(y^{p-1} g(y)/R^{p+1}\right) dy dR$$

$$= p^2 \int_0^{\|x-\theta\|} \int_{\|x-\theta\|}^{\infty} \left(g(y) y^{p-1}/R^{p+1}\right) dR dy$$

$$+ p^2 \int_{\|x-\theta\|}^{\infty} \int_y^{\infty} \left(g(y) y^{p-1}/R^{p+1}\right) dR dy$$

$$= (p/\|x-\theta\|^p) \int_0^{\|x-\theta\|} g(y) y^{p-1} dy$$

$$+ p \int_{\|x-\theta\|}^{\infty} \left(g(y)/y\right) dy$$

$$= \left(cH(\|x-\theta\|)\right)/(\|x-\theta\|^p) + p \int_{\|x-\theta\|}^{\infty} \left(g(y)/y\right) dy .$$

Returning to (3.1.4), we see that cancellation leads to

$$\int_{\|x-\theta\|}^{\infty} c(R) dF(R) = \int c(R) I_S(x,R) dF(R)$$

$$= g(\|x-\theta\|) .$$

The proof is now complete.

3.2. Estimators with smaller risks than the risk of one observation on a spherically symmetric unimodal distribution. Let X be a p x 1 random vector with a density with respect to Lebesgue measure given by

$$g(\|x-\theta\|) = \int c(R) I_S(x,R) dF(R) \text{ where } F(\cdot) \text{ is a known cdf} \tag{3.2.1}$$
$$\text{on } (0, \infty) \text{ and } c(R) \text{ and } I_S(x,R) \text{ are defined in (1.4).}$$

According to Theorem 3.1.1, X has a spherically symmetric unimodal distribution about θ.

Hence, since the density of X is a mixture of spherical uniforms, we may consider the random vector $X|R$ to have a spherical uniform distribution

$$(X|R \sim U\{||X-\theta||^2 \leq R^2\}) .$$

Therefore, directly by Lemma 6.1.1,

$$E_0(||X||^{-2}) = E[E_0(||X||^{-2}|R)] = \big(p/(p-2)\big)E(R^{-2}) .$$

Consider $\delta_a(X)$ given by (2.1.1), $\delta_a(X) = \big(1 - (a/||X||^2)\big)X$, for

$0 \leq a \leq (2b_0)/E_0(||X||^{-2}) = (2c_0)/E(R^{-2})$

where

$$b_0 = \begin{cases} p/(p+2) & p \geq 4 \\ .48 & p = 3 \end{cases}$$

and

$$c_0 = \begin{cases} (p-2)/(p+2) & p \geq 4 \\ .16 & p = 3 \end{cases} .$$

With respect to quadratic loss (1.1), when $0 \leq a \leq (2c_0)/E(R^{-2})$

$$[R(X,\theta) - R\big(\delta_a(X),\theta\big)]/a$$

$$(3.2.2) = E_\theta [2X'(X-\theta)||X||^{-2} - a||X||^{-2}]$$

$$\geq E_\theta [2X'(X-\theta)||X||^{-2} - (2c_0/ER^{-2})||X||^{-2}] .$$

The minimaxity of $\delta_{2c_o R^2}(X)$ for the spherical uniform distribution (see Theorems 2.1.1 and 2.2.1) implies

$$\begin{aligned} & E_\theta[2X'(X-\theta)\|X\|^{-2} - 2c_o R^2\|X\|^{-2}] \\ (3.2.3) \quad & = E\Big[E_\theta[2X'(X-\theta)\|X\|^{-2} - 2c_o R^2\|X\|^{-2}|R]\Big] \\ & \geq 0 . \end{aligned}$$

Thus, if

$$\begin{aligned} & E_\theta\|X\|^{-2} - ER^{-2}E_\theta(R^2\|X\|^{-2}) \\ & = \text{cov}\Big(E_\theta(R^2\|X\|^{-2}|R),\ R^{-2}\Big) \\ & \leq 0, \end{aligned}$$

then

$$\begin{aligned} & E_\theta[2X'(X-\theta)\|X\|^{-2} - \big(2c_o/E(R^{-2})\big)\|X\|^{-2}] \\ & \geq E_\theta[2X'(X-\theta)\|X\|^{-2} - 2c_o R^2\|X\|^{-2}] , \end{aligned}$$

clearly implying, by (3.2.2) and (3.2.3) that $\delta_a(X)$ is minimax for $0 \leq a \leq \big(2c_o/E(R^{-2})\big)$.

However, $E_\theta(R^2\|X\|^{-2}|R) = E_{\theta(R)}(\|Z\|^{-2}|R)$ where $Z|R \sim U\{\|Z-\theta(R)\|^2 \leq 1\}$ and $\theta(R) = [\|\theta\|/R,0,0,\ldots,0]'$. Lemma 6.2.3 implies $\|Z\|^2$ is stochastically ordered in $\|\theta(R)\|$ and hence, $E_{\theta(R)}[\|Z\|^{-2}|R]$ is a non-increasing function of $\|\theta(R)\|$ (see Lehmann [14], pages 73-74).

Therefore, for fixed $\|\theta\|$, $E_{\theta(R)}[\|Z\|^{-2}|R]$ is a non-decreasing function of R and since R^{-2} is a non-increasing function of R,

$\mathrm{cov}\left(E_{\theta(R)}(\|Z\|^{-2}|R),\ R^{-2}\right) \leq 0$. We summarize this result in the following theorem.

Theorem 3.2.1: If X is one observation on a s.s.u. distribution about θ with a density given by (3.2.1) and $\delta_a(X)$ is defined by (2.1.1) then when $0 \leq a \leq (2b_o)/E_0(\|X\|^{-2})$ and $b_o = \left(p/(p+2)\right)$ when $p \geq 4$ and $b_o = .48$ when $p = 3$, as defined by (2.3.2), $\delta_a(X)$ is minimax provided $p \geq 3$, $E_0(\|X\|^{-2})$ is finite and the loss is sum of squared errors (1.1).

James and Stein [12] proved, for X one observation on a p-variate normal distribution with mean vector θ and covariance matrix the identity, that $\delta_a(X) = \left(1 - (a/\|X\|^2)\right)X$ given by (2.1.1), is minimax for $0 \leq a \leq 2(p-2) = 2/(E_0\|X\|^{-2})$.

In the normal case, the James-Stein class of estimators includes our class. However, for $p \geq 4$, the only estimators in the James-Stein class which are not in our class are for values of a such that

$$\left(2p/(p+2)\right)/E_0(\|X\|^{-2}) \leq a \leq 2/E_0(\|X\|^{-2}) .$$

Similar statements hold when comparing our results to those of Strawderman [17] given on "variance" mixtures of normal distributions.

Since $p/(p+2) \to 1$, as $p \to \infty$, our class of estimators is, in a sense, approaching the James-Stein class for large p. Additionally, the best estimator in the normal case occurs when $a = \left(1/E_0(\|X\|^{-2})\right)$ which is always in our class. Furthermore, our bounds on a are the best possible which can be obtained for the whole class of s.s.u. distributions when $p \geq 4$, since we have already seen in section 2.1, that our bounds are the best possible for the spherical uniform distribution.

3.3. A larger class of minimax estimators. In this section we will consider estimators, of the mean vector of a s.s.u. distribution, given by

$$\delta_{a,r}(X) = \left(1 - a\left(r(\|X\|^2)/\|X\|^2\right)\right) X .$$

In the following theorem we will present sufficient conditions for $\delta_{a,r}(X)$ to be minimax.

Theorem 3.3.1: If X is a single observation on a p-dimensional distribution of the form (3.2.1) and $\delta_{a,r}(X)$ is defined by (2.3.1), then provided:

1) $0 \le a \le (2b_o)/E_0(\|X\|^{-2})$ (b_o is given by (2.3.2)),
2) $r(\|X\|^2)$ is non-decreasing,
3) $r(\|X\|^2)/\|X\|^2$ is non-increasing, and
4) $E_0(\|X\|^{-2})$ is finite,

the risk of $\delta_{a,r}(X)$ dominates (is less than or equal to) the risk of X for $p \ge 3$ and quadratic loss (1.1).

Proof: For $0 \le r(\cdot) \le 1$ and $0 \le a \le (2b_o)/E_0(\|X\|^{-2}) = (2c_o)/E(R^{-2})$,

$$[R(X,\theta) - R\left(\delta_{a,r}(X),\theta\right)]/a$$

$$\text{(3.3.1)} \qquad \ge E_\theta\left[r(\|X\|^2)[\left(2X'(X-\theta)-a\right)\|X\|^{-2}]\right]$$

$$\ge E_\theta\left[r(\|X\|^2)[2X'(X-\theta)\|X\|^{-2} - \left(2c_o/E(R^{-2})\right)\|X\|^{-2}]\right]$$

where c_o is defined by (2.1.2). As we noted in the previous section, $X|R$ may be considered as a spherical uniform random vector. Hence, since Theorem 2.3.1 implies the minimaxity of $\delta_{2c_oR^2}(X)$ for the spherical unfirom distribution,

$$E_\theta\left[r(\|X\|^2)\,[2X'(X-\theta)\|X\|^{-2} - 2c_o R^2\|X\|^{-2}]\right]$$

$$(3.3.2) \qquad = E_\theta\left[E[r(\|X\|^2)\left(2X'(X-\theta)\|X\|^{-2} - 2c_o R^2\|X\|^{-2}\right)|R]\right]$$

$$\geq 0.$$

Hence, as in the proof of Theorem 3.2.1, (3.3.1) and (3.3.2) imply

$$R(X,\theta) - R\left(\delta_{a,r}(X),\theta\right) \geq 0$$

for $0 \leq a \leq (2c_o)/E(R^{-2})$, if

$$E_\theta\left(r(\|X\|^2)\|X\|^{-2}\right) - E(R^{-2})E_\theta\left(r(\|X\|^2)R^2\|X\|^{-2}\right)$$

$$= \operatorname{cov}\left(E_\theta\left(r(\|X\|^2)R^2\|X\|^{-2}|R\right),\ \ R^{-2}\right)$$

$$\leq 0.$$

Since R^{-2} is non-increasing in R, the proof will be complete if

$$E_\theta\left(r(\|X\|^2)R^2\|X\|^{-2}|R\right)$$

$$= E_{\theta(R)}\left(r(R\|Z\|^2)\|Z\|^{-2}|R\right)$$

is non-decreasing in R, where $Z = X/R$ and $\theta(R) = [\|\theta\|/R, 0, \ldots, 0]'$. Hence, $Z|R \sim U\{\|Z-\theta(R)\|^2 \leq 1\}$. By the properties of $r(\cdot)$ in the statement of this theorem (properties 2) and 3)) and the stochastic ordering of $\|Z\|^2$ in $\|\theta(R)\| = \|\theta\|/R$ (see Lemma 6.2.3), if $R_1 \leq R_2$ then

$$E_{\theta(R_1)}\left(r(R_1\|Z\|^2)\|Z\|^{-2}|R_1\right)$$

$$\leq E_{\theta(R_2)}\left(r(R_1\|Z\|^2)\|Z\|^{-2}|R_2\right)$$

$$\leq E_{\theta(R_2)}\left(r(R_2\|Z\|^2)\|Z\|^{-2}|R_2\right).$$

Thus, $E_{\theta(R)}\left(r(R\|Z\|^2)\|Z\|^{-2}|R\right)$ is non-decreasing in R, completing this proof.

4. Minimax estimators of the location parameter of a p-dimensional ($p \geq 3$) spherically symmetric unimodal distribution with respect to general quadratic loss. Throughout sections 2 and 3, the only loss function considered was quadratic loss given by (1.1). In this section, we will explicitly extend the results of sections 2 and 3 to the case of general quadratic loss given by (1.2).

4.1. Minimax estimators for the mean vector of a spherical uniform distribution with known radius. Analogous results to those given in section 2 for estimating the mean vector of a p-dimensional ($p \geq 3$), spherical uniform distribution are presented in this section when the loss is general quadratic loss, given, as in (1.2), by

$$L(\delta,\theta) = (\delta-\theta)'D(\delta-\theta)$$

where D is a p x p positive definite symmetric matrix.

Consider one observation X having a p-dimensional spherical uniform distribution with a density given by (1.4). Let $\delta_a(X)$ be defined by (2.1.1) i.e. $\delta_a(X) = \left(1 - (a/\|X\|^2)\right)X$. The loss throughout will be general quadratic loss.

We will prove that $\delta_a(X)$ is minimax when $0 \leq a \leq 2a_o\left((\text{trace } D/d_L) - 2\right)R^2$ where d_L = maximum eigenvalue of D and

$$(4.1.1) \qquad a_o = \begin{cases} 1/(p+2) & \text{for } p \geq 4 \\ .16 & \text{for } p = 3 \end{cases} .$$

Note that when D is the identity, $L(\delta,\theta)$ is just quadratic loss and the result we will prove coincides with those proven in sections 2.1 and 2.2.

Since X is minimax, $\delta_a(X)$ is minimax if the difference in risks, $R(X,\theta) - R\left(\delta_a(X),\theta\right)$, is non-negative for all θ.

With respect to general quadratic loss (1.2),

$$R(X,\theta) - R\left(\delta_a(X),\theta\right)$$

$$= E_\theta(X-\theta)'D(X-\theta) - E_\theta\left(X-\theta-(a/\|X\|^2)X\right)'D\left(X-\theta-(a/\|X\|^2)X\right)$$

$$= aE_\theta[2X'D(X-\theta)\|X\|^{-2} - a(X'DX)\|X\|^{-4}]$$

$$= aE_0[(2X'DX+2\theta'DX)\|X+\theta\|^{-2} - a(X'DX+2\theta'DX+\theta'D\theta)\|X+\theta\|^{-4}].$$

Since D is a positive definite symmetric matrix, there exists an orthogonal matrix Q such that $Q'DQ = D_1$ = the diagonal matrix whose entries along the diagonal are the eigenvalues $d_1, d_2, \ldots, d_p$ of D, (see Anderson [1], pages 338-341). If we transform X, letting $Z = Q'X$ and $\theta^* = Q'\theta$

$$R(X,\theta) - R\left(\delta_a(X),\theta\right) = aE_0[\left(2Z'D_1Z+2(\theta^*)'D_1Z\right)\|Z+\theta^*\|^{-2}]$$

$$- a^2E_0[\left(Z'DZ + 2(\theta^*)'D_1Z + (\theta^*)'D_1(\theta^*)\right)\|Z+\theta^*\|^{-4}]$$

$$= R^*(X,\theta^*) - R^*\left(\delta_a(X),\theta^*\right)$$

where R^* is the risk with respect to general quadratic loss (1.2) when $D = D_1$.

We may thus assume without loss of generality that D is this diagonal matrix.

Hence, since $X = [X_1, X_2, \ldots, X_p]'$ and $\theta = [\theta_1, \theta_2, \ldots, \theta_p]'$ then

$$R(X,\theta) - R\left(\delta_a(X),\theta\right) = aE_0\left[\sum_{i=1}^{p} d_i[(2X_i^2+2\theta_iX_i)\|X+\theta\|^{-2} - a(X_i+\theta_i)^2\|X+\theta\|^{-4}]\right].$$

Immediately from Lemma 6.1.7 when $r(\cdot) = 1$,

$$R(X,\theta) - R\left(\delta_a(X),\theta\right)$$

$$= \left(\theta' D\theta a / \left((p-1)\|\theta\|^2\right)\right)\Big[E_0\left[\left(2(p-1)(X_1^2+\|\theta\|X_1)-2\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}\right]$$

$$- aE_0\left[\left((p-1)(X_1+\|\theta\|)^2 - \|Y\|^2\right)\left((X_1+\|\theta\|)^2 + \|Y\|^2\right)^{-2}\right]\Big]$$

$$+ \left(\mathrm{trD}a/(p-1)\right)\Big[E_0\left[2\|Y\|^2\left((X_1+\|\theta\|)^2 + \|Y\|^2\right)^{-1}\right]$$

$$- aE_0\left[\|Y\|^2\left((X_1+\|\theta\|)^2 + \|Y\|^2\right)^{-2}\right]\Big]$$

where $\|Y\|^2 = \sum_{i=2}^{p} X_i^2$ and $\mathrm{trD} = \mathrm{traceD} = \sum_{i=1}^{p} d_i$.

From (6.1.6) and (6.1.8)

$$E_0\left[\left(2(p-1)(X_1^2+\|\theta\|X_1) - 2\|Y\|^2\right)\left((X_1+\|\theta\|)^2 + \|Y\|^2\right)^{-1}\right]$$

$$= 2(p-1)\left[1 - E_0\left(\|\theta\|(X_1+\|\theta\|)\right)\left((X_1+\|\theta\|)^2 + \|Y\|^2\right)^{-1}\right]$$

$$- 2pE_0\left[\|Y\|^2\left((X_1+\|\theta\|)^2 + \|Y\|^2\right)^{-1}\right]$$

$$= 4M \int_0^R \left((R^2-y^2)^{\frac{p-1}{2}} / d_{R,\|\theta\|}(y)\right)\left(R^4-3\|\theta\|^2R^2+4\|\theta\|^2(R^2-y^2)\right)dy$$

$$- 4M \int_0^R\left((R^2-y^2)^{\frac{p-1}{2}} / d_{R,\|\theta\|}(y)\right)\left(R^4-\|\theta\|^2R^2 + 2\|\theta\|^2(R^2-y^2)\right)dy$$

where, as in (2.1.6), $d_{R,\|\theta\|}(y) = (R^2-\|\theta\|^2)^2 + 4\|\theta\|^2(R^2-y^2)$ and

$M = [(2R^2/p) \int_0^R (R^2-y^2)^{\frac{p-3}{2}} dy]^{-1}$.

Therefore,

(4.1.3)
$$E_0\left[\left(2(p-1)(X_1^2+\|\theta\|X_1)-2\|Y\|^2\right)\left((X_1+\|\theta\|)^2 + \|Y\|^2\right)^{-1}\right]$$
$$= -8M\|\theta\|^2 \int_0^R \left(\left(y^2(R^2-y^2)^{\frac{p-1}{2}}\right)/d_{R,\|\theta\|}(y)\right)dy.$$

Lemma 6.1.8 states that $E_0[((p-1)(X_1+\|\theta\|)^2-\|Y\|^2)((X_1+\|\theta\|)^2+\|Y\|^2)^{-2}] \geq 0$ and clearly, (4.1.3) implies $E_0[(2(p-1)(X_1^2+\|\theta\|X_1)-2\|Y\|^2)((X_1+\|\theta\|)^2+\|Y\|^2)^{-1}] \leq 0$. Hence, if d_L = maximum eigenvalue of D, then

$$
\begin{aligned}
& R(X,\theta) - R(\delta_a(X),\theta) \\
& \geq (d_L a/(p-1)) \Big[E_0[(2(p-1)(X_1^2+\|\theta\|X_1)-2\|Y\|^2)((X_1+\|\theta\|)^2+\|Y\|^2)^{-1}] \\
& \qquad - aE_0[((p-1)(X_1+\|\theta\|)^2-\|Y\|^2)((X_1+\|\theta\|)^2+\|Y\|^2)^{-2}]\Big] \\
& \qquad + (\mathrm{tr}Da/(p-1))\Big[E_0[2\|Y\|^2((X_1+\|\theta\|)^2+\|Y\|^2)^{-1}] \\
& \qquad - aE_0[\|Y\|^2((X_1+\|\theta\|)^2+\|Y\|^2)^{-2}]\Big] \\
& = [R(X,\theta) - R(\delta_a(X),\theta)]^*.
\end{aligned}
\tag{4.1.4}
$$

If,

$$
b^*(\|\theta\|) = \frac{E_0[(d_L(2(p-1)(X_1^2+\|\theta\|X_1)-2\|Y\|^2) + 2\mathrm{tr}D\|Y\|^2)((X_1+\|\theta\|)^2+\|Y\|^2)^{-1}]}{E_0[(d_L((p-1)(X_1+\|\theta\|)^2 - \|Y\|^2) + \mathrm{tr}D\|Y\|^2)((X_1+\|\theta\|)^2+\|Y\|^2)^{-2}]}
$$

then (4.1.4) implies

$$
R(X,\theta) - R(\delta_a(X,\theta)) \geq [R(X,\theta) - R(\delta_a(X),\theta)]^* \geq 0 \text{ if } 0 \leq a \leq b^*(\|\theta\|).
$$

Writing $b^*(\|\theta\|)$ in integral form using (6.1.6) - (6.1.9) we obtain the following:

$b^*(\|\theta\|) = A(\|\theta\|)/B(\|\theta\|)$, where,

$$
\begin{aligned}
A(\|\theta\|) &= -8\|\theta\|^2 d_L \int_0^R (y^2(R^2-y^2)^{\frac{p-1}{2}}/d_{R,\|\theta\|}(y))\,dy \\
&\quad + (4\mathrm{tr}D/p)\int_0^R ((R^2-y^2)^{\frac{p-1}{2}}/d_{R,\|\theta\|}(y))(R^4-\|\theta\|^2R^2+2\|\theta\|^2(R^2-y^2))\,dy, \\
B(\|\theta\|) &= d_L\int_0^R ((R^2-y^2)^{\frac{p-3}{2}}/d_{R,\|\theta\|}(y))\,((1-p)R^4+(p-1)R^2\|\theta\|^2+(pR^2-(p-2)\|\theta\|^2)(R^2-y^2))\,dy \\
&\quad + (\mathrm{tr}D/(p-2))\int_0^R ((R^2-y^2)^{\frac{p-3}{2}}/d_{R,\|\theta\|}(y))((p-1)R^4+(1-p)R^2\|\theta\|^2+(p\|\theta\|^2-(p-2)R^2)(R^2-y^2))\,dy
\end{aligned}
$$

and $d_{R,\|\theta\|}(y)$ is defined by (2.1.6).

Simple calculations using (6.1.1) lead to

$$\lim_{\|\theta\|\to\infty} b^*(\|\theta\|) = \Big(2/(p+2)\Big)\Big((\mathrm{tr}D/d_L) - 2\Big) R^2.$$

Hence, if trace $D < 2d_L$, there does not exist a minimax estimator of the form (2.1.1) for $a \geq 0$.

We will now prove the following theorem:

Theorem 4.1.1: If X is a single observation on a p-dimensional spherical uniform distribution and $\delta_a(X)$ is given by (2.1.1), then with respect to general quadratic loss (1.2), the risk of $\delta_a(X)$ dominates (is less than or equal to) the risk of X when $p \geq 3$ provided

$$0 \leq a \leq 2a_o^*\Big((\mathrm{tr}D/d_L) - 2\Big)/E_O(\|X\|^{-2}) = 2a_o\Big((\mathrm{tr}D/d_L) - 2\Big)R^2$$

where

$$(4.1.5) \qquad a_o^* = \begin{cases} p/\Big((p+2)(p-2)\Big) & \text{for} \quad p \geq 4 \\ .48 & \text{for} \quad p = 3 \end{cases}$$

and a_o is given by (4.1.1), $\mathrm{tr}D = \mathrm{trace}D \geq 2d_L$ and d_L = maximum eigenvalue of D.

Proof: Suppose $\mathrm{tr}D/d_L = q$, therefore, $2 \leq q \leq p$. If

$$[R(X,\theta) - R\Big(\delta_{2a_o(q-2)R^2}(X),\theta\Big)]^*/\Big(2a_o(q-2)d_L R^2\Big) = \Delta_q ,$$

then for $0 \leq a \leq 2a_o(q-2)R^2$, by (4.1.4), $[R(X,\theta) - R\Big(\delta_a(X),\theta\Big)]^*/(ad_L) \geq \Delta_q$.

In addition, (4.1.4) clearly implies

$$(4.1.6) \qquad \begin{aligned} \Delta_q = {} & 1/(p-1)\Big[E_O[\Big(2(p-1)(X_1^2+\|\theta\|X_1)-2\|Y\|^2\Big)\Big((X_1+\|\theta\|)^2+\|Y\|^2\Big)^{-1}] \\ & - 2a_o(q-2)R^2E_O[\Big((p-1)(X_1+\|\theta\|)^2-\|Y\|^2\Big)\Big((X_1+\|\theta\|)^2+\|Y\|^2\Big)^{-2}]\Big] \\ & + q/(p-1)\Big[E_O[2\|Y\|^2\Big((X_1+\|\theta\|)^2+\|Y\|^2\Big)^{-1}] \\ & - 2a_o(q-2)R^2E_O[\|Y\|^2\Big((X_1+\|\theta\|)^2+\|Y\|^2\Big)^{-2}]\Big]. \end{aligned}$$

Since $R(X,\theta) - R\left(\delta_a(X),\theta\right) \geq [R(X,\theta) - R\left(\delta_a(X),\theta\right)]^* \geq ad_L\ \Delta_q$,

it is clear that the proof will be complete if $\Delta_q \geq 0$ for all $\|\theta\|$.

However, for fixed $\|\theta\|$,

$$(d^2/dq^2)(\Delta_q) = \left(-2R^2 a_o/(p-1)\right) E_0[\|Y\|^2 \left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}] \leq 0,$$

implying that for fixed $\|\theta\|$, Δ_q is a concave function of q. Since $2 \leq q \leq p$, the concavity of Δ_q implies $\Delta_q \geq \text{minimum}\ (\Delta_2, \Delta_p)$.

When $q = p$, $\Delta_p =$

$$2[1 - E_0\left(\|\theta\|(X_1+\|\theta\|)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}\right) - a_o(p-2)R^2 E_0\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}] .$$

Note first that by the definition of a_o, given by (4.1.1), $(p-2)a_o = c_o$, when c_o, as in (2.1.2), is given by $c_o = \begin{cases} (p-2)/(p+2) & \text{when } p \geq 4 \\ .16 & \text{when } p = 3 \end{cases}$.

Hence, by (2.1.4), Δ_p simply equals (difference in risks for $a=2c_oR^2)/(2c_oR^2)$ when the risk is quadratic loss (1.1). Clearly, Theorems 2.1.1 and 2.2.1 imply this is non-negative.

We will now show that Δ_2 is also non-negative.

As (4.1.6) implies,

$$\Delta_2 = \left(2/(p-1)\right)E_0[\left((p-1)(X_1^2+\|\theta\|X_1)+\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}] .$$

Substituting in the expressions for the expected values given in (6.1.6) and (6.1.8)

$$\Delta_2 \propto \int_0^R \left((R^2-y^2)^{\frac{p-1}{2}} / d_{R,\|\theta\|}(y)\right)[R^4 - (p+1)\|\theta\|^2R^2 + (p+2)\|\theta\|^2(R^2-y^2)]dy = \Delta_2^*.$$

We will show Δ_2^* is non-negative for two cases:

Case 1: $\|\theta\|^2 \leq (p/2)R^2$

With respect to the density

$$g_{p,||\theta||}(y) = \begin{cases} \left((R^2-y^2)^{\frac{p-1}{2}}/d_{R,||\theta||}(y)\right)\Big/\int_0^R \left((R^2-y^2)^{\frac{p-1}{2}}/d_{R,||\theta||}(y)\right)dy & \text{for } 0 \le y \le R \\ 0 & \text{elsewhere} \end{cases},$$

which according to Lemma 6.2.1 has MLR non-decreasing in y for $||\theta|| \le R$ and MLR non-increasing in y for $||\theta|| \ge R$,

$$\Delta_2^*/\int_0^R\left((R^2-y^2)^{\frac{p-1}{2}}/d_{R,||\theta||}(y)\right)dy = R^4 -(p+1)R^2||\theta|| +(p+2)||\theta||^2E_{||\theta||}(R^2-Y^2)$$

$$\ge R^4 -(p+1)R^2||\theta||^2 +(p+2)||\theta||^2E_R(R^2-Y^2) = R^4 -(p+1)R^2||\theta||^2 + \left((p+2)(p-1)/p\right)R^2||\theta||^2$$

$$= (R^2/p)[pR^2 - 2||\theta||^2] \ge 0.$$

Case 2: $||\theta||^2 \ge (p/2)R^2$

Using (6.1.3) and Lemma (6.1.5), we may rewrite Δ_2^* as follows:

$$\Delta_2^* = (\tfrac{1}{4})\left[(p+2)\int_0^R(R^2-y^2)^{\frac{p-1}{2}}dy -[(p-2)R^4+2pR^2||\theta||^2+(p+2)||\theta||^4]\int_0^R\left((R^2-y^2)^{\frac{p-1}{2}}/d_{R,||\theta||}(y)\right)dy\right]$$

$$= \int_0^R\left((R^2-y^2)^{\frac{p-1}{2}}/4b_0||\theta||^2\right)dy\left[(p+2)b_0||\theta||^2-[(p-2)R^2+(p+2)||\theta||^2]\sum_{i=0}^{\infty}(-1)^ib_i(R^2/||\theta||)^i\right]$$

where $b_i = [(p-2i)/(p+2i)]b_{i-1}$, $i = 0, 1, 2, \ldots.$

Applying simple calculations we obtain

$$\Delta_2^* \propto \sum_{i=0}^{\infty}(-1)^ic_i(R^2/||\theta||^2)^i, \text{ where } c_i = (p-2)b_{i+1} - (p+2)b_{i+2}$$

for $i = 0, 1, 2, \ldots$. Thus, $c_i = \left(4(i+1)p/\left(p+2(i+2)\right)\right)b_{i+1}$ and $c_{2i+1} < (p/2)c_{2i}$. Therefore, when $||\theta||^2 \ge (p/2)R^2$,

$$\sum_{i=0}^{\infty}(-1)^ic_i(R^2/||\theta||^2)^i = \sum_{i=0}^{\infty}(-1)^ic_{2i}(R^2/||\theta||^2)^{2i} - \sum_{i=0}^{\infty}(-1)^ic_{2i+1}(R^2/||\theta||^2)^{2i+1} \ge 0.$$

We have shown that $\Delta_2 \ge 0$ and $\Delta_p \ge 0$, therefore,

$\Delta_q \ge$ minimum $(\Delta_2, \Delta_p) \ge 0$.

The proof of the theorem is now complete.

We now expand this class of estimators by considering estimators Baranchik [2] considered, namely estimators of the form

$$\delta_{a,r}(X) = \left(1 - a\left(r(||X||^2)/||X||^2\right)\right)X,$$

as given by (2.3.1).

Theorem 4.1.2. If $X = [X_1, X_2, \ldots, X_p]'$ is a p x 1 random vector with a spherical uniform distribution and $\delta_{a,r}(X)$ is given by (2.3.1), then provided $r(||X||^2)$ is non-decreasing and $p \geq 3$, $\delta_{a,r}(X)$ is minimax with respect to general quadratic loss (1.2) when $0 \leq a \leq 2R^2 a_o\left((trD/d_L) - 2\right)$ $= 2R^2 a_o^*\left((trD/d_L) - 2\right)/E_0(||X||^{-2})$, where a_o and a_o^* are defined by (4.1.1) and (4.1.5) respectively, and traceD = trD $\geq 2d_L$ = 2 (maximum eigenvalue of D).

Proof: Since $r(||X||^2)$ is non-decreasing,

$$R(X,\theta) - R\left(\delta_{a,r}(X),\theta\right) = aE_\theta\left[r(||X||^2)[2X'D(X-\theta)||X||^{-2} - ar(||X||^2)(X'DX)||X||^{-4}]\right]$$

$$\geq aE_\theta\left[r(||X||^2)[2X'D(X-\theta)||X||^{-2} - a(X'DX)||X||^{-4}]\right] \qquad (4.1.7)$$

$$= aE_0 r(||X+\theta||^2)[2X'D(X+\theta)||X+\theta||^{-2} - a(X'DX+2\theta'DX+\theta'D\theta)||X+\theta||^{-4}].$$

Clearly, we may assume without loss of generality that D is a diagonal matrix. Immediately by (4.1.7), Lemma 6.1.7 and Lemma 6.1.2, if $||Y||^2 = \sum_{i=2}^{p} X_i^2$, then

$$[R(X,\theta) - R\left(\delta_{a,r}(X),\theta\right)]/a$$

$$\geq \left(\theta'D\theta/\left((p-1)||\theta||^2\right)\right)\Big[E_{||\theta||}[r(||X||^2)\left(2(p-1)(X_1^2 - ||\theta||X_1) - 2||Y||^2\right)||X||^{-2}]$$

$$- aE_{||\theta||}[r(||X||^2)\left((p-1)X_1^2 - ||Y||^2\right)||X||^{-4}]\Big] \qquad (4.1.8)$$

$$+ \left(trD/(p-1)\right)\left[E_{||\theta||}[2r(||X||^2)||Y||^2||X||^{-2}] - aE_{||\theta||}[r(||X||^2)||Y||^2||X||^{-4}]\right]$$

where $E_{||\theta||}$ denotes the expected value when $\theta = [||\theta||, 0, 0, \ldots 0]'$.

Suppose we define $E_{||\theta||}r(X)h(X)$ as follows:

$$(4.1.9)\qquad \begin{aligned} E_{||\theta||}\left(r(X)h(X)\right) &= E_{||\theta||}[r(||X||^2)\left(2(p-1)(X_1^2-||\theta||X_1)-2||Y||^2\right)||X||^{-2}] \\ &\quad - aE_{||\theta||}[r(||X||^2)\left((p-1)X_1^2 - ||Y||^2\right)||X||^{-4}] \end{aligned}$$

Clearly, then (4.1.8) is equivalent to

$$(4.1.10)\qquad \begin{aligned} &[R(X,\theta) - R\left(\delta_{a,r}(X),\theta\right)]/a \\ &\geq \left(\theta' D\theta/\left((p-1)||\theta||^2\right)\right)E_{||\theta||}\left(r(X)h(X)\right) \\ &\quad + \left(trD/(p-1)\right)\left[E_{||\theta||}[2r(||X||^2)||Y||^2\left(||X||^2-(a/2)\right)||X||^{-4}]\right]. \end{aligned}$$

Case 1: $E_{||\theta||}r(X)h(X) \geq r(a/2)E_{||\theta||}h(X)$

Since $r(\cdot)$ is non-decreasing,

$$E_{||\theta||}[2r(||X||^2)||Y||^2\left(||X||^2-(a/2)\right)||X||^{-4}] \geq r(a/2)E_{||\theta||}[2||Y||^2\left(||X||^2-(a/2)\right)||X||^{-4}].$$

Hence, clearly, by (4.1.10),

$$[R(X,\theta) - R\left(\delta_{a,r}(X),\theta\right)] \geq r(a/2)[R(X,\theta) - R\left(\delta_a(X),\theta\right)] \geq 0$$

when $0 \leq a \leq 2a_o^*\left((trD/d_L) - 2\right)/E_0(||X||^{-2})$ (see Theorem 4.1.1).

Case 2: $E_{||\theta||}\left(r(X)h(X)\right) \leq r(a/2)E_{||\theta||}h(X)$

By (4.1.3),

$$\begin{aligned} &E_{||\theta||}[\left(2(p-1)(X_1^2-||\theta||X_1)-2||Y||^2\right)||X||^{-2}] \\ &\quad = E_0[\left(2(p-1)(X_1^2+||\theta||X_1)-2||Y||^2\right)\left((X_1+||\theta||)^2+||Y||^2\right)^{-1}] \leq 0. \end{aligned}$$

Similarly, Lemma 6.1.8 states

$$E_{||\theta||}[\left((p-1)X_1^2-||Y||^2\right)||X||^{-4}] = E_0[\left((p-1)(X_1+||\theta||)^2-||Y||^2\right)\left((X_1+||\theta||)^2+||Y||^2\right)^{-2}]$$

is non-negative.

Hence, clearly by the definition of $E_{||\theta||}\left(r(X)h(X)\right)$ given by (4.1.9), $E_{||\theta||}\left(r(X)h(X)\right) \leq 0$. Hence, if d_L = maximum eigenvalue of D, then

$$
\begin{aligned}
&[R(X,\theta) - R\left(\delta_{a,r}(X),\theta\right)]/a \\
&\quad \geq \left(d_L/(p-1)\right)E_{||\theta||}\left(r(X)h(X)\right) \\
&\qquad + \left(\mathrm{tr}D/(p-1)\right)\left[E_{||\theta||}[2r(||X||^2)||Y||^2\left(||X||^2-(a/2)\right)||X||^{-4}]\right] \\
&\quad = [\left(R(X,\theta) - R\left(\delta_{a,r}(X),\theta\right)\right)/a]^*.
\end{aligned}
$$

For $\theta \leq a \leq 2R^2a_o\left((\mathrm{tr}D/d_L) \ -2\right)$, $\delta_{a,r}(X)$ will be minimax if $[\left(R(X,\theta) - R\left(\delta_{a,r}(X),\theta\right)\right)/a]^* \geq 0$ when $a = 2R^2a_o\left((\mathrm{tr}D/d_L) \ -2\right)$ or equivalently, if $\Delta_q\left(r(||X||^2)\right) \geq 0$, where $q = \mathrm{tr}D/d_L$, and $\Delta_q\left(r(||X||^2)\right) = (d_L)^{-1}[\left(R(X,\theta) - R\left(\delta_{a,r}(X),\theta\right)\right)/a]^*$ when $a = 2R^2a_o(q-2)$.

Hence,

$$
\begin{aligned}
\Delta_q\left(r(||X||^2)\right) = {} & 1/(p-1)\Big[E_{||\theta||}[r(||X||^2)\left(2(p-1)(X_1^2-||\theta||X_1)-2||Y||^2\right)||X||^{-2}] \\
& - 2a_oR^2(q-2)E_{||\theta||}[r(||X||^2)\left((p-1)X_1^2-||Y||^2\right)||X||^{-4}]\Big] \\
& +\left(2q/(p-1)\right)\Big[E_{||\theta||}[r(||X||^2)||Y||^2||X||^{-2}]-a_oR^2(q-2)E_{||\theta||}[r(||X||^2)||Y||^2||X||^{-4}]\Big].
\end{aligned}
\tag{4.1.11}
$$

For fixed $||\theta||$, $\Delta_q\left(r(||X||^2)\right)$ is a concave function of q clearly implying that $\Delta_q\left(r(||X||^2)\right) \geq \text{minimum}\left(\Delta_2\left(r(||X||^2)\right), \Delta_p\left(r(||X||^2)\right)\right)$. If we show that $\Delta_2\left(r(||X||^2)\right) \geq 0$ and $\Delta_p\left(r(||X||^2)\right) \geq 0$ then $\Delta_q\left(r(||X||^2)\right) \geq 0$ for $2 \leq q \leq p$, and the proof will be complete.

Allowing $q = p$ in (4.1.11) and since $2a_oR^2(p-2) = 2c_oR^2$, where c_o is given by (2.1.2), we have $\Delta_p\left(r(||X||^2)\right) = 2E_{||\theta||}[r(||X||^2)\left(1 - (||\theta||X_1+c_oR^2)||X||^{-2}\right)] \geq 0$, as proven in Theorem 2.3.1.

Before proceeding, note, that when $r(\cdot) \equiv 1$, a shift of X to the origin, clearly shows that $\Delta_2\left(r(\|X\|^2)\right) = \Delta_2(1) = \Delta_2$, where Δ_2 is given by (4.1.6) when $q = 2$. In the proof of Theorem 4.1.1, Δ_2 was shown to be non-negative.

Subcase 2.1: $\|\theta\|^2 \geq R^2$

From (4.1.11), $\Delta_2\left(r(\|X\|^2)\right) = (2/(p-1)E)_{\|\theta\|}[r(\|X\|^2)\left((p-1)(X_1^2-\|\theta\|X_1)+\|Y\|^2\right)\|X\|^{-2}]$.

According to Lemma 6.1.9,

$$E_{\|\theta\|}[r(\|X\|^2)(X_1^2\|X\|^{-2})] = \left((p-1)/p\right)E_{\|\theta\|}[r(\|X\|^2)\left((\|X\|^2-R^2+\|\theta\|^2)/2\|\theta\|\right)(X_1\|X\|^{-2})$$
$$+ (1/p)E_{\|\theta\|}r(\|X\|^2) \ .$$

Hence,

$$\begin{aligned}\Delta_2\left(r(\|X\|^2)\right) = (2/p)[2E_{\|\theta\|}r(\|X\|^2)&+\left((p-2)/2\|\theta\|\right)E_{\|\theta\|}\left(r(\|X\|^2)X_1\right) \\ &- \left(\left((p-2)R^2+(p+2)\|\theta\|^2\right)/2\|\theta\|\right)E_{\|\theta\|}\left(r(\|X\|^2)(X_1\|X\|^{-2})\right)] .\end{aligned} \tag{4.1.12}$$

By Lemma 6.2.4, $E_{\|\theta\|}(X_1 \big| \|X\|^2)$ is non-decreasing in $\|X\|^2$ when $\|\theta\| \geq R$. Moreover, Lemma 6.2.2 implies $E_{\|\theta\|}(X_1\|X\|^{-2}\big|\|X\|^2)$ is non-increasing in $\|X\|^2$ when $\|\theta\| \geq R$. Combining these two facts with the assumption that $r(\|X\|^2)$ is a non-decreasing function of $\|X\|^2$, (4.1.12) implies

$$\Delta_2\left(r(\|X\|^2)\right) \geq [E_{\|\theta\|}r(\|X\|^2)]\ \Delta_2 \ .$$

As we noted earlier, Δ_2 is non-negative, hence the proof is complete for this subcase.

Subcase 2.2: $0 \leq \|\theta\|^2 \leq 4a_o R^2$

From (4.1.11), we may rewrite $\Delta_2\left(r(\|X\|^2)\right)$ as follows:

$$\begin{aligned}\Delta_2\left(r(\|X\|^2)\right) = \left(2/(p-1)\right)\Big[E_{\|\theta\|}r(\|X\|^2) &- E_{\|\theta\|}[r(\|X\|^2)(\|\theta\|X_1)\|X\|^{-2}] \\ &+ (p-2)E_{\|\theta\|}[r(\|X\|^2)(X_1^2-\|\theta\|X_1)\|X\|^{-2}]\Big] .\end{aligned} \tag{4.1.13}$$

For $||\theta||^2 \le 4a_oR^2$,

$$E_{||\theta||}[r(||X||^2)(X_1^2-||\theta||X_1||X||^{-2}] \ge -(||\theta||^2/4)E_{||\theta||}[r(||X||^2)||X||^{-2}]$$

$$\ge -a_oR^2E_{||\theta||}[r(||X||^2)||X||^{-2}] .$$

Thus, clearly, by (4.1.13),

$$\Delta_2\left(r(||X||^2)\right) \ge \left(1/(p-1)\right) \Delta_p\left(r(||X||^2)\right) \ge 0.$$

Subcase 2.3: $4a_oR^2 \le ||\theta||^2 \le R^2$

Suppose $E_{||\theta||}[r(||X||^2)(||\theta||X_1)||X||^{-2}] \ge E_{||\theta||}[r(||X||^2)(||\theta||^2-a_oR^2)||X||^{-2}]$

then $E_{||\theta||}[r(||X||^2)(X_1^2-||\theta||X_1)||X||^{-2}] \ge E_{||\theta||}[r(||X||^2)(||\theta||X_1-||\theta||^2)||X||^{-2}]$

$$\ge -a_oR^2E_{||\theta||}[r(||X||^2)||X||^{-2}] ,$$

and so,

$$\Delta_2\left(r(||X||^2)\right) \ge \left(1/(p-1)\right)\Delta_p\left(r(||X||^2)\right) \ge 0.$$

To complete the proof, we will now show that $\Delta_2\left(r(||X||^2)\right) \ge 0$ if $E_{||\theta||}[r(||X||^2)(||\theta||X_1)||X||^{-2}] \le E_{||\theta||}[r(||X||^2)(||\theta||^2-a_oR^2)||X||^{-2}]$.

Lemma 6.2.5 states that $E_{||\theta||}[r(||X||^2)(||X||^2-R^2-||\theta||^2)/2||\theta||)(X_1||X||^{-2})]$

$$\ge -cE_{||\theta||}[r(||X||^2)(||\theta||X_1)||X||^{-2}]$$

when $||\theta||^2 \ge \left((2/cp) - 1\right)R^2$.

Substituting in $c = \left(2a_o/(1-a_o)\right)$, for $\left((1-(p+1)a_o)/pa_o\right)R^2 \le ||\theta||^2 \le R^2$, we have that

$$E_{||\theta||}[r(||X||^2)\left((||X||^2-R^2-||\theta||^2)/2||\theta||\right)(X_1||X||^{-2})]$$

$$\ge -\left(2a_o/(1-a_o)\right)E_{||\theta||}[r(||X||^2)(||\theta||X_1)||X||^{-2}]$$

$$\ge -\left(2a_o/(1-a_o)\right)E_{||\theta||}[r(||X||^2)(||\theta||^2-a_oR^2)||X||^{-2}]$$

$$\ge -2a_oR^2E_{||\theta||}[r(||X||^2)||X||^{-2}] .$$

By rewriting (4.1.12) we obtain

$$(4.1.14)\quad \begin{aligned}\Delta_2\left(r(\|X\|^2)\right) = (2/p)\Big[&2E_{\|\theta\|}r(\|X\|^2)-2\|\theta\|E_{\|\theta\|}[r(\|X\|^2)X_1\|X\|^{-2}] \\ &+ (p-2)E_{\|\theta\|}[r(\|X\|^2)\left((\|X\|^2-\|\theta\|^2-R^2)/2\|\theta\|\right)(X_1\|X\|^{-2})]\Big].\end{aligned}$$

Using (4.1.14), we see that

$$\begin{aligned}\Delta_2\left(r(\|X\|^2)\right) &\geq (2/p)\Big[2E_{\|\theta\|}r(\|X\|^2)-2\|\theta\|E_{\|\theta\|}[r(\|X\|^2)X_1\|X\|^{-2}] \\ &\qquad - 2a_o(p-2)R^2E_{\|\theta\|}[r(\|X\|^2)\|X\|^{-2}]\Big] \\ &= (2/p)\left(\Delta_p\left(r(\|X\|^2)\right)\right) \geq 0.\end{aligned}$$

When $p \geq 4$, $\left((1-(p+1)a_o)/pa_o\right) = (1/p) \leq 4a_o$ and when $p = 3$, $\left((1-(p+1)a_o)/pa_o\right) = (3/4) \geq 4a_o = .64$. Hence, for $p \geq 4$, $\Delta_2\left(r(\|X\|^2)\right) \geq 0$ when $R^2/p \leq \|\theta\|^2 \leq R^2$ implying $\Delta_2\left(r(\|X\|^2)\right) \geq 0$ when $4a_oR^2 \leq \|\theta\|^2 \leq R^2$.

For $p = 3$, $\Delta_2\left(r(\|X\|^2)\right) \geq 0$ when $(3/4)R^2 \leq \|\theta\|^2 \leq R^2$. We must, therefore, show that when $p = 3$, $\Delta_2\left(r(\|X\|^2)\right) \geq 0$ for $.64R^2 \leq \|\theta\|^2 \leq .75R^2$.

Proceeding as before, if $c = \left(8a_o/(3-4a_o)\right)$, calculations using the inequality obtained from Lemma 6.2.5 imply that when $(11/48)R^2 \leq \|\theta\|^2 \leq (3/4)R^2$,

$$\begin{aligned}&E_{\|\theta\|}[r(\|X\|^2)\left((\|X\|^2-R^2-\|\theta\|^2)/2\|\theta\|\right)(X_1\|X\|^{-2})] \\ &\quad \geq -\left(8a_o/(3-4a_o)\right)E_{\|\theta\|}[r(\|X\|^2)(\|\theta\|^2-a_oR^2)\|X\|^{-2}] \\ &\quad \geq -2a_oR^2E_{\|\theta\|}[r(\|X\|^2)\|X\|^{-2}].\end{aligned}$$

Hence, using expression (4.1.14), $\Delta_2\left(r(\|X\|^2)\right) \geq (2/p)\Delta_p\left(r(\|X\|^2)\right) \geq 0$ when $(11/48)R^2 \leq \|\theta\|^2 \leq (3/4)R^2$. Since $11/48 < .64$, this implies

$$\Delta_2\left(r(\|X\|^2)\right) \geq 0 \quad \text{for } .64R^2 \leq \|\theta\|^2 \leq .75R^2.$$

Hence, for $p \geq 3$ and for all $\|\theta\|$, $\Delta_2\left(r(\|X\|^2)\right) \geq 0$ and $\Delta_p\left(r(\|X\|^2)\right) \geq 0$ thus implying $\Delta_q\left(r(\|X\|^2)\right) \geq \text{minimum}\left(\Delta_2\left(r(\|X\|^2)\right), \Delta_p\left(r(\|X\|^2)\right)\right) \geq 0.$

The proof of the theorem is complete.

4.2. Minimax estimators of the mean vector of a spherically symmetric unimodal distribution. In this section we consider X one observation on a p-dimensional spherically symmetric unimodal (s.s.u.) distribution. Hence, as in (3.2.1), the density of X is given by

$$g(\|x-\theta\|) = \int c(R) I_S(x,R) dF(R)$$

where $F(\cdot)$ is a known cdf on $(0, \infty)$ and $c(R)$ and $I_S(X,R)$ are defined in (1.4).

We will show that with respect to general quadratic loss (1.2), $\delta_a(X) = \left(1-(a/\|X\|^2)\right) X$ is minimax when $0 \leq a \leq \left(2a_o/E(R^{-2})\right)\left((\mathrm{tr}D/d_L)-2\right)$ where a_o is defined by (4.1.1), d_L = maximum eigenvalue of D and $\mathrm{tr}D = \mathrm{trace}D \geq 2d_L$.

Since we may view the random vector $X|R$ as having a spherical uniform distribution $(X|R \sim U\{\|X-\theta\|^2 \leq R^2\})$, if we proceed exactly as we did in Section 4.1, clearly, we obtain the same inequality $\left((4.1.4)\right)$ for $R(X,\theta) - R\left(\delta_a(X),\theta\right)$ as we did for the uniform distribution. That is,

$$\begin{aligned}
R(X,\theta) - R\left(\delta_a(X),\theta\right) &= aE\left[E_\theta[2X'D(X-\theta)\|X\|^{-2}-a(X'DX)\|X\|^{-4}|R]\right] \\
&\geq \left(d_L a/(p-1)\right)\Big[E_0[\left(2(p-1)(X_1^2+\|\theta\|X_1)-2\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}] \\
&\quad - aE_0[\left((p-1)(X_1+\|\theta\|)^2-\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}]\Big] \\
&\quad + \left(\mathrm{tr}Da/(p-1)\right)\Big[E_0[2\|Y\|^2\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}] \\
&\quad - aE_0[\|Y\|^2\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}]\Big] \\
&= [R(X,\theta) - R\left(\delta_a(X),\theta\right)]^*.
\end{aligned}$$

If $q = \mathrm{tr}D/d_L$, $2 \leq q \leq p$, and $\Delta_q^* = [R(X,\theta) - R\left(\delta_a(X),\theta\right)]^*/ad_L$ when $a = 2a_o(q-2)/E(R^{-2})$,

then

$$\begin{aligned}\Delta_q^* = \left(1/(p-1)\right)\Big[&E_0[\left(2(p-1)(X_1^2+\|\theta\|X_1)-2\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}] \\ &-\left(2a_o(q-2)/E(R^{-2})\right)\ E_0[\left((p-1)(X_1+\|\theta\|)^2-\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}]\Big] \\ &+\left(q/(p-1)\right)\Big[E_0[2\|Y\|^2\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}] \\ &-\left(2a_o(q-2)/E(R^{-2})\right)E_0[\|Y\|^2\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}]\Big],\end{aligned}$$

and the risk of $\delta_a(X)$ is less than or equal to the risk of X for $0 \leq a \leq 2a_o(q-2)/E(R^{-2})$ if $\Delta_q^* \geq 0$. For fixed $\|\theta\|$, Δ_q^* is a concave function of q and since $2 \leq q \leq p$, it follows that $\Delta_q^* \geq$ minimum (Δ_2^*, Δ_p^*).

However, when $q = p$, $2a_o(q-2) = 2c_o$, where c_o is defined by (2.1.2) and $\Delta_p^* = \left(E_\theta\|X-\theta\|^2 - E_\theta\|\delta_a(X)-\theta\|^2\right)/a$ for $a = 2c_o/E(R^{-2})$. Hence, by Theorem 2.3.1, $\Delta_p^* \geq 0$.

When $q = 2$

$$\Delta_2^* = \left(2/(p-1)\right)E_0[\left((p-1)(X_1^2+\|\theta\|X_1)+\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}].$$

When $q = 2$ and X has a spherical uniform distribution, the above expectation is just Δ_2, defined by (4.1.6) which, for that case was proven to be non-negative in the proof of Theorem 4.1.2. Hence, since $X|R \sim U\{\|X-\theta\|^2 \leq R^2\}$, it is clear that $\Delta_2^* \geq 0$.

We have just proven that $\delta_a(X)$ given by (2.1.1) is minimax with respect to general quadratic loss for $0 \leq a \leq \left(2a_o/E(R^{-2})\right)\left((\mathrm{tr}D/d_L)-2\right)$. In the following theorem we will formally state this result.

Theorem 4.2.1: If X is a p x 1 random vector ($p \geq 3$) with a density given by (3.2.1), then with respect to general quadratic loss (1.2), $\delta_a(X)$ given by (2.1.1) is minimax provided

$$0 \le a \le \left(2a_o/E(R^{-2})\right)\left((\mathrm{tr}D/d_L)-2\right) = 2a_o^*\left((\mathrm{tr}D/d_L)-2\right)/E_0(\|X\|^{-2}),$$

(4.2.1) where a_o and a_o^* are defined by (4.1.1) and (4.1.5) respectively, and $\mathrm{tr}D \ge 2d_L = 2$ (maximum eigenvalue of D).

We next expand this class of estimators.

Theorem 4.2.2: If X is one observation on p-dimensional s.s.u. distribution about θ with a density given by (3.2.1) and $\delta_{a,r}(X) = \left(1-a\left(r(\|X\|^2)/\|X\|^2\right)\right)X$, as defined by (2.3.1), then provided $p \ge 3$, $\left(r(\|X\|^2)/\|X\|^2\right)$ is non-increasing, $r(\|X\|^2)$ is non-decreasing and (4.2.1) is satisfied, then the risk of $\delta_{a,r}(X)$ dominates the risk of X with respect to general quadratic loss (1.2).

Proof: Since $X|R$ may be considered to be a spherical uniform random vector, we proceed exactly as we did in the proof of Theorem 4.1.2. Hence, we see that the theorem will be proven if $\Delta_q^*\left(r(\|X\|^2)\right)$ is non-negative for $2 \le q = (\mathrm{tr}D/d_L) \le p$ and

$$\begin{aligned}\Delta_q^*\left(r(\|X\|^2)\right) = \left(1/(p-1)\right)\Big[&E_{\|\theta\|}\left[r(\|X\|^2)\left[\left(2(p-1)(X_1^2-\|\theta\|X_1)-2\|Y\|^2\right)\|X\|^{-2}\right]\right] \\ &- \left(2(q-2)a_o/E(R^{-2})\right)E_{\|\theta\|}\left[r(\|X\|^2)\left[\left((p-1)X_1^2-\|Y\|^2\right)\|X\|^{-4}\right]\right]\Big] \\ &+ \left(q/(p-1)\right)\Big[E_{\|\theta\|}[r(\|X\|^2)\ 2\|Y\|^2\|X\|^{-2}] \\ &- \left(2(q-2)a_o/E(R^{-2})\right)E_{\|\theta\|}[r(\|X\|^2)\|Y\|^2\|X\|^{-4}]\Big] .\end{aligned}$$

Clearly, $\Delta_q^*\left(r(\|X\|^2)\right)$ is a concave function of q for fixed $\|\theta\|$.

For $q = p$, $2(q-2)a_o = 2c_o$, where c_o is defined by (2.1.2) and hence, by Theorem 3.3.2,

$$\Delta_p^* = E_{\|\theta\|}\left[r(\|X\|^2)\left[2-\left(2\|\theta\|X_1+\left(2c_o/E(R^{-2})\right)\right)\|X\|^{-2}\right]\right] \ge 0.$$

Moreover, since $X|R \sim U\{\|X-\theta\|^2 \le R^2\}$, the proof of Theorem 4.1.2 immediately shows that $\Delta_2^*\left(r(\|X\|^2)\right) \ge 0$.

Hence, by the concavity of $\Delta_q^*(r(\|X\|^2))$, $\Delta_q^*(r(\|X\|^2)) \geq \text{minimum}\left(\Delta_2^*(r(\|X\|^2)),\ \Delta_p^*(r(\|X\|^2))\right) \geq 0$, thus completing the proof of this theorem.

5. Remarks and conclusions. We conclude with some observations on the multiple observation case and a discussion of the merits of using these improved estimators.

Consider n observations, $X_1, X_2, \ldots, X_n$, on a p-dimensional s.s.u. distribution about θ. For the normal distribution the problem is easily reduced by sufficiency to one observation, $\bar{X}$. However, this is not the case here. Pitman's estimator given by $\delta(X_1, X_2, \ldots, X_n) = X_1 - E_0[X_1 | Y_2, Y_3, \ldots, Y_n]$ where $Y_i = X_i - X_i$, $i = 2, 3, \ldots, n$, is the best invariant estimator and hence, there exist estimators which are better than it. For the normal distribution, Pitman's estimator is $\bar{X}$ and in fact, in one dimension when $n \geq 3$, Pitman's estimator being $\bar{X}$ characterizes the normal distribution (see Kagan, Linnik and Rao [13], chapter 7). In general, for $n \geq 3$, Pitman's estimator is not $\bar{X}$. It is clear from the definition of Pitman's estimator that it is $\bar{X}$ for $n = 1, 2$.

If the distribution of Pitman's estimator is spherically symmetric unimodal about θ, the problem is reduced to the case of one observation. We investigated this question and, as yet, have only proven that Pitman's estimator has a s.s.u. distribution about θ when sampling from a spherical uniform distribution about θ. The more general problem is still under investigation. Note that $\bar{X}$, which is a convolution of random vectors having s.s.u. distributions has a s.s.u. distribution. Hence, we may use the estimators of sections 2 - 4 to improve on $\bar{X}$ with respect to quadratic and general quadratic loss.

We now return to the case when X is one observation on a p-dimensional s.s.u. distribution about θ. Consider the estimator $\delta_a^+(X) = \max\left(0, \left(1-(a/\|X\|^2)\right)\right)X$. It is clear by the work of Baranchik [2] that $\delta_a^+(X)$ is better than $\delta_a(X) = \left(1-(a/\|X\|^2)\right)X$ with respect to quadratic loss (1.1). For $0 \leq a \leq (2b_o)/E_0(\|X\|^{-2})$, where $b_o = \left(p/(p+2)\right)$ when $p \geq 4$ and $b_o = .48$ when $p = 3$, $\delta_a(X)$ and hence $\delta_a^+(X)$ is better than X (see Theorem 3.2.1). These new estimators are certainly not very difficult to calculate and the improvements over X, in some instances can be very large. Consider, for example, the case of estimating θ, when X has a p-dimensional spherical uniform distribution about θ, with respect to quadratic loss (1.1). According to Lemma 6.1.1, the risk of X equals $E_\theta(\|X-\theta\|^2) = (p/R^p)\int_0^R r^{p+1}dr = \left(p/(p+2)\right)R^2$, for all θ. Again, by Lemma 6.1.1, when $\theta = [0, 0, \ldots, 0]'$, $R\left(\delta_a(X),0\right) = E_0(\|X\|^2) - a[2-aE_0(\|X\|^{-2})]$ $= \left(p/(p+2)\right)R^2 - a[2-a\left(p/(p-2)\right)R^{-2}]$. Since $R\left(\delta_a(X),0\right)$ is a convex function of a and for $p \geq 4$, $0 \leq a \leq (2b_o)/E_0(\|X\|^{-2}) = 2\left((p-2)/(p+2)\right)R^2$, $R\left(\delta_a(X),0\right)$ $\geq R\left(\delta_{((p-2)/p)R^2}(X), 0\right) = \left(4/\left(p(p+2)\right)\right)R^2$. Clearly, this risk becomes very small as p becomes large. Moreover, $R\left(\delta_{((p-2)/p)R^2}(X), 0\right)/R(X,0) = (4/p^2)$. Therefore, for $p \geq 4$, the risk of $\delta_a(X)$ when $a = \left((p-2)/p\right)R^2$ is at most ($\frac{1}{4}$) of the risk of X at the origin.

If we now wish further improvement, we may consider $\delta_a^+(X)$. When $\|X\|^2 \geq a$, $\delta_a^+(X) = \delta_a(X)$ and when $\|X\|^2 \leq a$, $\delta_a^+(X) = 0$. Since the risks only depend on $\theta = [\|\theta\|, 0, 0, \ldots, 0]'$, Lemma 6.1.6, implies that when $\|\theta\| \geq R + \sqrt{a}$, $\|X\|^2$ is always greater than or equal to a, and thus, $\delta_a^+(X)$ coincides with $\delta_a(X)$. Similarly, if $\|\theta\| \leq \sqrt{a} - R$ then $\|X\|^2$ is always less than or equal to a and thus $\delta_a^+(X) = 0$. Therefore, for $p \geq 6$ and $R^2 \leq a \leq 2\left((p-2)/(p+2)\right)R^2$, $R\left(\delta_a^+(X),0\right) = 0$.

We thus see that when X is one observation on a p-dimensional, $(p \geq 4)$, spherical uniform distribution, there exists an a, $0 \leq a \leq \left(2(p-2)/(p+2)\right)R^2$, for which $\delta_a^+(X)$ improves over X for all $\|\theta\|$ with a large improvement at the origin.

Moreover, when $p \geq 6$, there exists an a for which $\delta_a^{+}(X)$ is minimax and $R\left(\delta_a^{+}(X),0\right) = 0$ with respect to quadratic loss (1.1). Since these new improved estimators do not present any difficulties in calculation, they may easily be used in place of X for estimating the mean of a spherical uniform and more generally, the mean of a s.s.u. distribution.

We complete this section with a note on the robustness of these estimators. Our improved estimators are robust in the sense that for any p-dimensional, $(p \geq 3)$, distribution about θ satisfying $E_0(\|X\|^{-2}) \leq c$, where c is a given constant, $\delta_a(X)$ is minimax with respect to general quadratic loss (1.2) for any a such that $0 \leq a \leq 2a_o^*\left((trD/d_L) - 2\right)/c$ where, as in (4.1.5), $a_o^* = \left(p/\left((p+2)(p-2)\right)\right)$ for $p \geq 4$ and $a_o^* = .48$ for $p = 3$ and $trD > 2d_L$. Hence, we need not always know exactly what $E_0(\|X\|^{-2})$ in order to use these improved estimators.

It is hoped that these results add some insight into the behavior of the James-Stein type estimators and with this, perhaps those using these estimators will do so more confidently.

6. Appendix

6.1. Integral expressions, expectations and density derivations. In this section we present useful integral evaluations and various densities as well as many expected values which aid in the calculation of important expressions. Many integrals were evaluated with the aid of Burington [8].

$$(6.1.1)\qquad \int_0^R (R^2-y^2)^q dy = (2q/2q+1)R^2 \int_0^R (R^2-y^2)^{q-1} dy$$

$$(6.1.2)\qquad \int_0^R (R^2-y^2)^{\frac{1}{2}} dy = (\pi R^2/4)$$

$$(6.1.3)\qquad \begin{aligned} &4\|\theta\|^2 \int_0^R \left((R^2-y^2)^q / d_{R,\|\theta\|}(y)\right) dy \\ &= \int_0^R (R^2-y^2)^{q-1} dy - (R^2-\|\theta\|^2)^2 \int_0^R \left((R^2-y^2)^{q-1}/d_{R,\|\theta\|}(y)\right) dy \end{aligned}$$

where

(6.1.4) $d_{R,||\theta||}(y) = (R^2-||\theta||)^2 + 4||\theta||^2(R^2-y^2)$.

Lemma 6.1.1: If X is a p x 1 random vector with a spherical uniform distribution ($X \sim U\{||X - \theta||^2 \leq R^2\}$) then for any integrable function $g(||X-\theta||)$, $E_\theta g(||X-\theta||) = M_o\int_0^R r^{p-1}g(r)dr$, where $M_o = p/R^p$. In particular, $E_0(||X||^{-2}) = \big(p/(p-2)\big)R^{-2}$ and $E_0(||X||^{-1}) = \big(p/(p-1)\big)R^{-1}$.

Proof: Results follow by a conversion to spherical coordinates.

Lemma 6.1.2: If $X = [X_1, X_2, \ldots, X_p]' \sim U\{||X-\theta||^2 \leq R^2\}$ then $E_\theta g(X'\theta, ||X||^2) = E_{||\theta||}g(X_1||\theta||,||X||^2)$ where $E_{||\theta||}$ denotes the expected value when $\theta = [||\theta||, 0, \ldots, 0]'$. Moreover, if $\theta = [||\theta||, 0, \ldots, 0]'$,

$$E_{||\theta||}g(X_1||\theta||, ||X||^2, ||Y||^2) = E_0 g\big(||\theta||(X_1+||\theta||), (X_1+||\theta||)^2+||Y||^2, ||Y||^2\big)$$

$$= M\int_0^R\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r^{p-2}g\big(||\theta||(x_1+||\theta||), (x_1+||\theta||)^2+r^2, r^2\big)dx_1\,dr$$

where $||Y||^2 = \sum_{i=2}^{p} X_i^2$ and

(6.1.5) $M = [(2/p)\int_0^R (R^2-y^2)^{\frac{p-3}{2}}\,dy]^{-1}$.

Proof: The first part is true, with a simple transformation of variables by applying the p x p orthogonal matrix P which is such that $P\theta = [||\theta||, 0, \ldots, 0]$.

The second part is easily obtained by translating to the origin and then transforming $(x_2, x_3, \ldots, x_p)$ into spherical coordinates.

Lemma 6.1.3: If $X = [X_1, X_2, \ldots, X_p]'$ has a spherical uniform distribution then

(6.1.6)
$$E_\theta\big(2X'(X-\theta)||X||^{-2}\big) = 2-2E_0[||\theta||(X_1+||\theta||)\big((X_1+||\theta||)^2+||Y||^2\big)^{-1}]$$

$$= \big(4M/(p-1)\big)\int_0^R\big((R^2-y^2)^{\frac{p-1}{2}}/d_{R,||\theta||}(y)\big)[R^4-3||\theta||^2R^2+4||\theta||^2(R^2-y^2)]dy$$

and

$$E_\theta(\|X\|^{-2}) = E_0\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}$$

(6.1.7)

$$= \left(2M/(p-2)\right)\int_0^R \left((R^2-y^2)^{\frac{p-3}{2}}/d_{R,\|\theta\|}(y)\right)[R^4-\|\theta\|^2R^2+2\|\theta\|^2(R^2-y^2)]dy$$

where $\|Y\|^2 = \sum_{i=2}^{p} X_i^2$, $d_{R,\|\theta\|}(y)$ is defined by (6.1.4) and M is defined by (6.1.5).

Proof: We obtain the desired results by applying Lemma 6.1.2, integrating with respect to x_1 and then integrating by parts and transforming variables to $y = \sqrt{R^2-r^2}$.

Lemma 6.1.4: If $X = [X_1, X_2, \ldots, X_p]' \sim U\{\|X-\theta\|^2 \le R^2\}$ and $\theta = [\|\theta\|, 0, \ldots, 0]'$, then

$$E_{\|\theta\|}(\|Y\|^2\|X\|^{-2}) = E_0\left(\|Y\|^2\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}\right)$$

(6.1.8)

$$= (2M/p)\int_0^R\left((R^2-y^2)^{\frac{p-1}{2}}/d_{R,\|\theta\|}(y)\right)[R^4-\|\theta\|^2R^2+2\|\theta\|^2(R^2-y^2)]dy$$

and

$$E_{\|\theta\|}(\|Y\|^2\|X\|^{-4}) = E_0\left(\|Y\|^2\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}\right)$$

(6.1.9)

$$= \left(M/(p-2)\right)\int_0^R\left((R^2-y^2)^{\frac{p-3}{2}}/d_{R,\|\theta\|}(y)\right)[(p-1)R^2-(p-1)\|\theta\|^2R^2 +(p\|\theta\|^2-(p-2)R^2)(R^2-y^2)]dy$$

where $\|Y\|^2 = \sum_{i=2}^{p} X_i^2$ and $d_{R,\|\theta\|}(y)$ and M are defined by (6.1.4) and (6.1.5) respectively.

Lemma 6.1.5: When $p \geq 3$,

$$\int_0^R \left((R^2-y^2)^{\frac{p-3}{2}} / \left(d_{R,\|\theta\|}(y)\right)\right) dy$$

$$= \begin{cases} \left(\int_0^R (R^2-y^2)^{\frac{p-3}{2}} dy\right)[h(\|\theta\|,R)]_p & \text{when } \|\theta\| \geq R \\ \left(\int_0^R (R^2-y^2)^{\frac{p-3}{2}} dy\right)[h(R,\|\theta\|)]_p & \text{when } \|\theta\| \leq R \end{cases}$$

where

$$(6.1.10) \qquad [h(\|\theta\|,R)]_p = \left(a_0\|\theta\|^2(R^2+\|\theta\|^2)\right)^{-1} \sum_{i=0}^{\infty} a_i(R^2/\|\theta\|^2)^i$$

and

$$(6.1.11) \qquad [h(R,\|\theta\|)]_p = \left(a_0R^2(R^2+\|\theta\|^2)\right)^{-1} \sum_{i=0}^{\infty} a_i(\|\theta\|^2/R^2)^i$$

and $a_i = [\left(p - 2(i+1)\right)/\left(p + 2(i-1)\right)]a_{i-1}$ for $i = 0, 1, 2, \ldots.$

and $d_{R,\|\theta\|}(y)$ is defined by (6.1.4).

Proof: The proof is a proof by induction by first assuming the lemma is true for p and utilizing (6.1.3) to prove it is true for $p + 2$. In order to prove the lemma for $p \geq 3$, it is straightforward to show it is true for $p = 3$ and $p = 4$.

Lemma 6.1.6: Suppose $X = [X_1, X_2, \ldots, X_p]' \sim U\{\|X-\theta\|^2 \leq R^2\}$ and $\theta = [\|\theta\|, 0, 0, \ldots, 0]'$. If $Z = \|X\|^2$,

$S_1 = \left((x_1,z): \left((z-R^2+\|\theta\|^2)/2\|\theta\|\right) \leq x_1 \leq \sqrt{z},\ (R-\|\theta\|)^2 \leq z \leq (R+\|\theta\|)^2\right)$ and

$S_2 = \left((x_1,z): -\sqrt{z} \leq x_1 \leq \sqrt{z},\ 0 \leq z \leq (R-\|\theta\|)^2\right)$ then the joint density of X_1 and Z is given by

(6.1.2) $$f_{\|\theta\|}(x_1,z) = (M/2)(z-x_1^2)^{\frac{p-3}{2}} I_{S_1}(x_1,z) \quad \text{when} \quad \|\theta\| \geq R$$

and

(6.1.3) $$f_{\|\theta\|}(x_1,z) = (M/2)(z-x_1^2)^{\frac{p-3}{2}} I_{S_1 \cup S_2}(x_1,z) \quad \text{when} \quad \|\theta\| \leq R$$

where M is given by (6.1.5) and $I_S(x_1,R)$ and $I_{S_1 \cup S_2}(x_1,R)$ are the indicator functions over the sets S_1 and $(S_1 \cup S_2)$, respectively.

Proof: We obtain these densities by taking $(d/dx_1)(d/dz)P_{\|\theta\|}(X_1 \leq x_1, \|X\|^2 \leq z)$.

Lemma 6.1.7: If $X = [X_1, X_2, \ldots, X_p]' \sim U\{\|X-\theta\|^2 \leq R^2\}$ where $\theta = [\theta_1, \theta_2, \ldots, \theta_p]'$ and $\|Y\|^2 = \sum_{i=2}^{p} X_i^2$, then for any integrable function $r(\cdot)$

$$(p-1)E_0\left[r(\|X+\theta\|^2)[(2X_i^2+2\theta_i X_i)\|X+\theta\|^{-2} - a(X_i+\theta_i)^2\|X+\theta\|^{-4}]\right]$$

$$= (\theta_i^2/\|\theta\|^2)\Bigg[E_0\left[r\left((X_1+\|\theta\|)^2+\|Y\|^2\right)[\left(2(p-1)(X_1^2+\|\theta\|X_1)-2\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}]\right]$$

$$- aE_0\left[r\left((X_1+\|\theta\|)^2+\|Y\|^2\right)[\left((p-1)(X_1+\|\theta\|)^2-\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}]\right]\Bigg]$$

$$+ 2E_0[r\left((X_1+\|\theta\|)^2+\|Y\|^2\right)\|Y\|^2\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-1}]$$

$$- aE_0[r\left((X_1+\|\theta\|)^2+\|Y\|^2\right)\|Y\|^2\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}].$$

Proof: We obtain the desired result by making two transformation of variables. If P is a (p-1) x (p-1) orthogonal transformation such that $P[\theta_1, \theta_2, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_p]' = [\|\theta\|_i, 0, \ldots, 0]'$ where $\|\theta\|_i = \sqrt{\|\theta\|^2-\theta_i^2}$, we first transform to $s = [s_2, s_3, \ldots, s_p]' = P[x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p]'$.

There exists an orthogonal transformation Q such that $Q[\theta_i, \|\theta\|_i, 0, \ldots, 0]' = [\|\theta\|, 0, \ldots, 0]'$, and if $z = [z_1, z_2, \ldots, z_p]' = Q[x_i, s_2, \ldots, s_p]'$ then $x_i = (\theta_i/\|\theta\|)z_1-(\|\theta\|_i/\|\theta\|)z_2$, $s_2 = (\|\theta\|_i/\|\theta\|)z_1 + (\theta_i/\|\theta\|)z_2$ and $s_i = z_i$ for $i \geq 3$. With this transformation and using the fact that for any constant c,

$$\int_{(\|z\|^2 \leq R^2)} r\left((z_1+\|\theta\|)^2+\|Y\|^2\right)\left(z_2(\|\theta\|+cz_1)\right)\left((z_1+\|\theta\|)^2+\|Y\|^2\right)^{-1} dz = 0$$

we obtain the desired result.

Lemma 6.1.8: If $X = [X_1, X_2, \ldots, X_p]' \sim U\{\|X-\theta\|^2 \leq R^2\}$ and $\|Y\|^2 = \sum_{i=2}^{p} X_i^2$ then $E_0[\left((p-1)(X_1+\|\theta\|)^2-\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}] \geq 0$ when $p \geq 3$.

Proof: Using (6.1.9) and Lemma 6.1.5 it can be shown that

$$E_0[\|Y\|^2\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}]$$

$$\propto \begin{cases} g_1(\|\theta\|,R) & \text{when } \|\theta\| \geq R \\ g_2(R,\|\theta\|) & \text{when } \|\theta\| \leq R \end{cases}$$

where

$$g_1(\|\theta\|,R) = (pR^2-p\|\theta\|^2)\left([h(\|\theta\|,R)]_p\right) + \left(p\|\theta\|^2-(p-2)R^2\right)\left([h(\|\theta\|,R)]_{p+2}\right)$$

and

$$g_2(R,\|\theta\|) = (pR^2-p\|\theta\|^2)\left([h(R,\|\theta\|)]_p\right) + \left(p\|\theta\|^2-(p-2)R^2\right)\left([h(R,\|\theta\|)]_{p+2}\right).$$

Moreover, by (6.1.7), (6.1.9), and Lemma 6.1.5

$$E_0[\left((p-1)(X_1+\|\theta\|)^2-\|Y\|^2\right)\left((X_1+\|\theta\|)^2+\|Y\|^2\right)^{-2}$$

$$\propto \begin{cases} g_1(R,\|\theta\|) & \text{when } R \geq \|\theta\| \\ g_2(\|\theta\|,R) & \text{when } R \leq \|\theta\| \end{cases}.$$

Hence, it is non-negative for all $\|\theta\|$. Q.E.D.

Lemma 6.1.9: If $X = [X_1, X_2, \ldots, X_p]' \sim U\{\|X-\theta\|^2 \leq R^2\}$, $\theta = [\|\theta\|, 0, \ldots, 0]'$, and $r(\|X\|^2)$ is any integrable function, then

$$E_{\|\theta\|}[r(\|X\|^2)(X_1^2\|X\|^{-2})] = (p-1/p)E_{\|\theta\|}r(\|X\|^2)\left((\|X\|^2-R^2+\|\theta\|^2)/2\|\theta\|\right)(X_1\|X\|^{-2})$$

$$+ (1/p)E_{\|\theta\|}r(\|X\|^2).$$

Proof: Use of the expressions for the joint density of X_1 and $||X||^2$ given in Lemma 6.1.6 and an integration by parts completes the proof.

6.2. Auxillary lemmas. In this section we present lemmas which contain important properties which aid in the proof of theorems in sections 2 - 4.

Lemma 6.2.1: If Y is a random variable with a density with respect to Lebesgue measure given by

$$g_{2q+1}(y) = \begin{cases} \left((R^2-y^2)^q/d_{R,||\theta||}(y)\right) \Big/ \int_0^R \left((R^2-y^2)^q/d_{R,||\theta||}(y)\right)dy & \text{when } 0 \le y \le R \\ 0 & \text{elsewhere} \end{cases}$$

where, as in (6.1.5), $d_{R,||\theta||}(y) = (R^2-||\theta||^2)^2 + 4||\theta||^2(R^2-y^2)$, then the distribution of Y has monotone likelihood ratio (MLR) non-decreasing in Y when $||\theta|| \le R$ and MLR non-increasing in Y when $||\theta|| \ge R$.

Proof: It is straightforward to show that for $0 \le ||\theta||_1 \le ||\theta||_2 \le R$, $(d/dy)\left(g_{2q+1,||\theta||_2}(y)/g_{2q+1,||\theta||_1}(y)\right) \ge 0$ and for $R \le ||\theta||_1 \le ||\theta||_2$, $(d/dy)\left(g_{2q+1,||\theta||_2}(y)/g_{2q+1,||\theta||_1}(y)\right) \le 0$, which completes the proof.

Lemma 6.2.2: If $X = [X_1, X_2, \ldots, X_p]' \sim U\{||X-\theta||^2 \le R^2\}$ and $Z = ||X||^2$, then for any c, and for fixed $||\theta||$ satisfying $||\theta||^2 \ge (1-2c)R^2$, $E_{||\theta||}\left((||\theta||X_1+cR^2)\,||X||^{-2}\,\middle|\,||X||^2\right)$ is non-increasing in $||X||^2$.

Proof: If $S = (||\theta||X_1+cR^2)||X||^{-2}$, it can be shown that $f_\theta(s|z) = h_z(s)$ has MLR non-increasing in s for each fixed $||\theta||$ for which $||\theta||^2 \ge (1-2cR^2)$. Hence, $E_Z(S)$ is non-increasing in Z (see Lehmann [15], page 74). Q.E.D.

Lemma 6.2.3: If X has a p-dimensional spherical uniform distribution about θ, then $P_\theta\{\|X\|^2 \geq c\} = P_{\|\theta\|}\{\|X\|^2 \geq c\}$ is a non-decreasing function of $\|\theta\|$.

Proof: Suppose $\theta_1 = [\|\theta_1\|, 0, \ldots, 0]'$ and $\theta_2 = [\|\theta_2\|, 0, \ldots, 0]'$ and $\|\theta_1\| \leq \|\theta_2\|$.

Case 1: $\|\theta_1\|^2 \geq R^2$ or $(\|\theta_2\|^2 \leq R^2$ and $c \geq R^2)$

$$P_{\|\theta_1\|}(\|X\|^2 \geq c) = P_{\|\theta_1\|}\left(\|X+(\theta_2-\theta_1)\|^2 \geq c\right) \geq P_{\|\theta_1\|}(\|X\|^2 \geq c) \text{ since}$$

$\|X\|^2 \geq c$ implies $\|X+(\theta_2-\theta_1)\|^2 \geq c$.

Case 2: $\|\theta_2\|^2 \leq R^2$ and $c \leq R^2$

For this case, $\|X-\theta_2\|^2 \leq R^2$ implies $\|X-\theta_1\|^2 \leq R^2$ when $\|X\|^2 \leq c$ and hence $P_{\|\theta_2\|}(\|X\|^2 \leq c) \leq P_{\|\theta_1\|}(\|X\|^2 \leq c)$.

Case 3: $\|\theta_1\|^2 \leq R^2 \leq \|\theta_2\|^2$

Cases 1 and 2 imply

$$P_{\|\theta_2\|}(\|X\|^2 \geq c) \geq P_R(\|X\|^2 \geq c) \geq P_{\|\theta_1\|}(\|X\|^2 \geq c).$$

The proof is now complete.

Lemma 6.2.4: If $X = [X_1, X_2, \ldots, X_p]' \sim U\{\|X-\theta\|^2 \leq R^2\}$, $\theta = [\|\theta\|, 0, 0, \ldots, 0]'$ and $p \geq 3$ then for fixed $\|\theta\|$ satisfying $\|\theta\| \geq R$, $E_{\|\theta\|}(X_1 \big| \|X\|^2)$ is non-decreasing in $\|X\|^2$.

Proof: The density $f_{\|\theta\|}(x_1|z) = g_z(x_1)$, where $z = \|x\|^2$, has MLR non-decreasing in x_1 for fixed $\|\theta\| \geq R$. Hence, $E_Z(X_1)$ is non-decreasing in Z. Q.E.D.

Lemma 6.2.5: If $X = [X_1, X_2, \ldots, X_p]' \sim U\{\|X-\theta\|^2 \leq R^2\}, \theta = [\|\theta\|, 0, 0, \ldots, 0]'$, then for $p \geq 3$, c a positive constant, and $\|\theta\|^2 \geq \left((2/cp)-1\right)R^2$,

$$E_{\|\theta\|}\left[r(\|X\|^2)\left[\left((\|X\|^2-R^2-\|\theta\|^2)/2\|\theta\|\right)(X_1\|X\|^{-2})\right]\right] \geq -cE_{\|\theta\|}[r(\|X\|^2)(\|\theta\|X_1\|X\|^{-2})]$$

where $r(\|X\|^2)$ is a non-decreasing function.

Proof: If $g(z) = \left(z - R^2 - (1-2c)\|\theta\|^2\right)/2\|\theta\|$ where $Z = \|X\|^2$, we will prove that when $\|\theta\|^2 \geq \left((2/cp) - 1\right)R^2$, $E_{\|\theta\|}[r(Z)g(Z)(X_1Z^{-1})] \geq 0$. Using the joint density for X_1 and Z given by Lemma 6.1.6, we have

$$E_{\|\theta\|}[r(Z_1)g(Z_1)(X_1Z_1^{-1})] \geq r\left(\left(R^2+(1-c)\|\theta\|^2\right)/2\|\theta\|\right)E_{\|\theta\|}[g(Z)X_1Z_1^{-1}].$$

Expression (6.1.6) implies

$$E_{\|\theta\|}[g(Z)(X_1Z^{-1})] \propto -(R^2-2c\|\theta\|^2)(R^2-\|\theta\|^2)^2+\left(R^2+(1-2c)\|\theta\|^2\right)(R^4-3\|\theta\|^2R^2)$$
$$+ 4\|\theta\|^4E_{\|\theta\|}(R^2-Y^2)$$

where $E_{\|\theta\|}(R^2-Y^2)$ is the expected value with respect to the density $g_{p,\|\theta\|}(y)$ given by (6.2.1). Using the MLR properties of $g_{p,\|\theta\|}(y)$ given in Theorem 6.2.1, we have $E_{\|\theta\|}(R^2-Y^2) \geq E_R(R^2-Y^2) = \left((p-1)/p\right)R^2$ which leads to the desired result. Q.E.D.

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